Time-reversal properties in the coupling of quantum angular momenta

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After a synoptic panorama about some still unsolved foundational problems involving time-reversal, we show that the *double time-reversal superselection rule* of Nonrelativistic Quantum Mechanics is redundant.

We then analyze which, among the symmetries of the Clebsch-Gordan coefficients, may be inferred through time-reversal considerations.

Finally, we show how Coupling Theory allows to improve our comprehension of the fact that, in presence of hidden symmetries, Wigner's Theorem concerning Kramers degeneration cannot be applied, by analyzing a set of Z uncoupled massive particles, both in the case in which they are bosons of spin zero and in the case in which they are fermions of spin $\frac{1}{2}$, subjected to a Keplerian force's field and considering the coupling of the 2Z quantum angular momenta resulting by the hidden SO(4) symmetry owed to the fact that, apart from the rotational SO(3) symmetry responsible of the conservation of the angular momentum, the Laplace-Runge-Lenz vector is also conserved.

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I. INTRODUCTION: A SYNOPTICAL PANORAMA OF SOME STILL UNSOLVED FOUNDATIONAL PROBLEMS INVOLVING TIME-REVERSAL AND MENU OF THIS PAPER

The philosophical, mathematical and physical investigations concerning the nature of Time have been pursued by the human kind for almost three thousand years.

What is time ?

The most common answer to this question given by contemporary scientists (and particularly those with an underlying strongly positivistic attitude) is that:

1. for Science time is the physical observable operationally defined by the clock.
2. every other speculation is a (not particularly serious) metaphysical intellectual masturbation not belonging to Science.

Let us observe that, instead of definitely closing the issue, such an answer opens a Pandora's box simply translating the original question into the new question:

What is a clock ?

whose not-triviality is linked to a plethora of reasons whose analysis goes beyond the purposes of this paper.

We will limit ourselves to remind that:

1. despite the existence of atomic clocks that, using the principle of the Maser, realize the more precise measurements of time that our actual technology allows (cfr. for instance the 6^{th} chapter "Two-State Systems, Principle of the Maser" of [1]), the many theorems stating in different words the impossibility of defining a time operator in Nonrelativistic Quantum Mechanics (cfr. for instance [2] and [3], the section 12.7 "The measurement of time" and the section 12.8 "Time and energy complementarity" of [4] as well as [5]) invalidates the conceptual possibility of defining a genuine quantum clock.
2. the impossibility in General Relativity of extending globally local clocks (cfr. the section 2.4.4 "Meanings of time" of [6]).

Actually we strongly think that the philosophical issue about the nature of time is so deeply important for Physics that we cannot leave it in the hands of bad philosophers [1].

Of course the answer about the nature of time will be available only when we will know the real Laws of Nature conciliating General Relativity and Quantum Mechanics, of which we possess nowadays a plethora of candidates whose Popper's falsifiability exists only as a matter of principle, most of the corresponding phenomenology being detectable only at the Planck's scale (lengths of the order of the Planck' length $l_P := \sqrt{\frac{\hbar G}{c^3}} \approx 10^{-33}\, cm$, energies of the order of Planck's energy $E_P := \sqrt{\frac{\hbar c}{G}} \approx 10^{19}\, Gev$, times of the order of Planck's time $t_P := \sqrt{\frac{\hbar G}{c^5}} \approx 10^{-44}\, s$) that is enormously far from our actual possibility of experimental investigation (see for instance [8]).

In this paper we will remain at a lower level taking into account only Non Relativistic Quantum Mechanics that is a very good approximation of the true Laws of Nature for enough weak gravitational fields and for objects moving at velocities enough smaller than the velocity of light.

At such a level, only two notions of time, among the many mentioned by Rovelli, will be pertinent to our analysis: the Newtonian absolute time and the thermodynamical time.

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[1] The different philosophies of Time can be roughly divided in two classes, according to whether they consider Time as an entity pertaining the Subject (as in Augoustine's or, more radically, in Immanuel Kant's thought in which the Object is the result of the categorical synthesis of the Subject's intellectual structure) or the Object. It has to be remarked, with this regard, that the Philosophy of Science underlying the Scientific Method and properly formalized by Karl Popper's Falsification Theory (overcoming the unphysical feature of Rudolf Carnap's Verification Theory consisting in that it would require an infinite number of measurements to verify a theory) is, as to the basic epistemological issue concerning the nature of truth, essentially a restyling of the Scholastic viewpoint *"veritas est adaequatio rei et intellectus"*. Within such a philosophical framework, if at a foundational level time exists (it is curious, with this regard that the idea that at a Quantum Gravity level time doesn't exist, considered as one of the possibilities in the 6^{th} section "Timeless Interpretation of Quantum Gravity" of [3] and explicitly advocated by Carlo Rovelli in [6], [7] coincides with the basic philosophical statement of John Ellis Mc Taggart's "proved" through very involute and baroque speculations) it is an objective property, i.e. an entity belonging to the Object of Knowledge. Contemporary fashionable post-modern philosophies adhering, to some degree, to Friedrich Nietsche's nihilism and/or to the essential point of Martin Heidegger's thought, namely that the knowledge that the Subject has of the Object is conditioned by a pre-knowledge determined by a temporally-dependent information's horizon into which he is thrown, giving up the solid ground of Scientific Reason, open the door to the more dangerous intellectual (and ethical) irrationalisms.

These two notions conflict since Nonrelativistic Classical Mechanics is symmetric under reversal of the Newtonian absolute time, while the Second Principle states that Thermodynamics is not invariant under the reversal of the Newtonian absolute time:

given a system of N particles leaving in the physical three-dimensional euclidean space $(\mathbb{R}^3, \delta := \delta_{\mu\nu} dx^\mu \otimes dx^\nu)$ and hence having hamiltonian $H : \mathbb{R}^{6N} \mapsto \mathbb{R}$:

$$H(\vec{q}_1, \vec{p}_1, \cdots, \vec{q}_N, \vec{p}_N) \;:=\; \sum_{i=1}^{N} \frac{|p_i|_\delta^2}{2m_i} \,+\, \sum_{i<j=1}^{N} V_{interaction}(\vec{q}_i, \vec{q}_j) \,+\, \sum_{i=1}^{N} V_{external}(\vec{q}_i) \tag{1.1}$$

[2] and defined the time-reversal operator as the map:

$$t \;\overset{T}{\mapsto}\; t' := -t \tag{1.2}$$

it follows that:

$$(\vec{q}_1, \vec{p}_1, \cdots, \vec{q}_N, \vec{p}_N)(t) \overset{T}{\mapsto} (\vec{q}_1, -\vec{p}_1, \cdots, \vec{q}_N, -\vec{p}_N)(t) \tag{1.3}$$

and hence the Hamilton's equations:

$$\begin{cases} \dot{\vec{p}_i} \;=\; -\partial_{\vec{q}_i} H \\ \dot{\vec{q}_i} \;=\; \partial_{\vec{p}_i} H \end{cases} \tag{1.4}$$

are T-invariant.

As a consequence, if $c : [t_{initial}, t_{final}] \mapsto \mathbb{R}^{6N}$ is the solution of the Cauchy problem consisting of equations 1.4 and the initial-conditions:

$$\begin{cases} \vec{q}_i(t_{initial}) = \vec{q}_i^{initial} \in \mathbb{R}^3 \\ \vec{p}_i(t_{initial}) = \vec{p}_i^{initial} \in \mathbb{R}^3 \end{cases} \tag{1.5}$$

and introduced:

$$\begin{cases} \vec{q}_i^{final} \;:=\; \vec{q}_i(t_{final}) \\ \vec{p}_i^{final} \;:=\; \vec{p}_i(t_{final}) \end{cases} \tag{1.6}$$

then $c^{-1}(t) := c(t_{final} - t)$ will be the solution of the Cauchy problem consisting of equations 1.4 with the initial conditions:

$$\begin{pmatrix} \vec{q}_i(t_{initial}) \\ \vec{p}_i(t_{initial}) \end{pmatrix} \;=\; T \begin{pmatrix} \vec{q}_i^{final} \\ \vec{p}_i^{final} \end{pmatrix} \;=\; \begin{pmatrix} \vec{q}_i^{final} \\ -\vec{p}_i^{final} \end{pmatrix} \tag{1.7}$$

Contrary the Second Law of Thermodynamics holding for any isolated thermodynamical system:

$$\frac{dS_{thermodynamic}}{dt} \;\geq\; 0 \tag{1.8}$$

is clearly not T-invariant.

It is common opinion in the Physics' scientific community (see for instance [9], [10], [11]) that Ludwig Boltzmann solved the (apparent) paradox through his invention of Statistical Mechanics, by showing that a probabilistic description of a coarse-grained representation of the reversible [3] microscopic dynamics (on which some suitable ergodicity or chaoticity condition is eventually added [12]) of an high number of microscopic objects may generate the phenomenological irreversibility observed macroscopically.

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[2] where of course, given a vector $\vec{v}$, we have used the notation $|v|_\delta^2 := v^\mu \delta_{\mu\nu} v^\nu = \vec{v}^2$ to emphasize its dependence from the underlying euclidean geometry.

[3] Let us remark that the term "reversible" is often used in the literature, particularly in that devoted to "Irreversible Quantum Dynamics", oscillating alternatively between the following three different meanings:

first meaning: reversibility as invariance under time-reversal.

second meaning: a dynamics is called reversible if it doesn't comport an increase of the thermodynamical entropy of the Universe.

third meaning: a map is called reversible if it is injective.

In this paper we will always use the term "reversible" with the first meaning, while, when we want to refer to the second meaning, we will use the locution "thermodynamical reversibility".

Indeed the expression for the thermodynamical entropy obtained using the microcanonical ensemble:

$$S(E,N,V) \ := \ k_{Boltzmann} \log(\frac{\int_{\Lambda} \prod_{i=1}^{N} d\vec{q}_{i} d\vec{p}_{i} \delta(H-E)}{N!}) \tag{1.9}$$

(where of course $\int_{\Lambda} \prod_{i=1}^{N} d\vec{q}_{i} d\vec{p}_{i} \ = V$) [4] summarizes the basic Boltzmann's idea according to which:

1. the thermodynamical entropy is directly proportional to the logarithm of the number of microstates corresponding to the given thermodynamical macrostate.
2. the Second Law of Thermodynamical expresses the fact that an isolated system evolves from ordered macrostates (i.e. macrostates to which correspond few microstates) to disordered macrostates (i.e. macrostates to which correspond many microstates) [5].

Anyway the objections that Josef Loschmidt, Henri Poincaré and Ernst Zermelo (see for instance the 2^{th} chapter "Time Reversal in Classical Mechanics" of [13], the 5^{th} chapter "Time irreversibility and the H-theorem" of [14] and [15]) moved to Boltzmann's viewpoint are, in our modest opinion, still disturbing:

1. the argument according to which the operation of acting through the time-reversal operator could be only performed by a Maxwell's demon acting on single particles and is hence irrelevant for a statistical approach to the system doesn't take into account the following basic fact: that in many experimental situations we are nowadays technologically able to construct Maxwell'demons acting, according to some programmed algorithm, on individual particles of a many-particle system and that, in this case, the *probabilistic approach to information* has to be considered jointly with the *algorithmic approach to information* [6] by taking into account the contribution of the *algorithmic information* of the demon to the thermodynamical entropy (see the 3^{th} section "Perpetual motion" of the 6^{th} chapter "Statistical Entropy" of [18], [19], [20], [21], the 8^{th} chapter "Physics, Information and Computation" of [22] and the 5^{th} chapter "Reversible Computation and the Thermodynamics of Computing" of [23]).
2. Poincare's Recurrence Theorem (see for instance the Lecture I "Measurable Transformations, Invariant Measures, Ergodic Theorems" of [24]) states that the microscopic dynamics will return arbitrarily close to the initial conditions.

   In every course of Statistical Mechanics (see for instance the 10^{th} chapter "Irreversibility and the Approach to Equilibrium" of [25]) it is taught that this doesn't conflict with the thermodynamical irreversibility since an explicit computation of the *recurrence's times* shows that they are always enormously big (of many orders of magnitude greater than the age of the Universe) and is, in particular, of many orders of magnitude greater than the duration of the time interval required to reach the thermodynamical equilibrium.

   Such an answer is not, in our modest opinion, conceptually satisfying from a foundational viewpoint, since it implies that the existence of the thermodynamical time-asymmetry (i.e., adopting a commonly used locution, the thermodynamical *arrow of time*) would be only a transient phenomenon:

   obviously, from a mathematical viewpoint, the density of any time interval $[0, T_{recurrence}]$ is zero also if $T_{recurrence}$ is, to use of the words of [12], *"beyond eternity"* (i.e. of many orders of magnitude greater than the age of the Universe):

$$\lim_{t \rightarrow +\infty} \frac{\mu_{Lebesgue}([0,T_{recurrence}])}{\mu_{Lebesgue}([0,t])} \ = \ \lim_{t \rightarrow +\infty} \frac{T_{recurrence}}{t} \ = \ 0 \ \ \forall T_{recurrence} \in (0, +\infty) \tag{1.10}$$

   But we are too much convinced of the validity and importance of the Second Principle of Thermodynamics to accept a viewpoint according to which the existence of the *thermodynamical arrow* would be only a transient phenomenon [7].

[4] beside the factor N! whose theoretical justification can be given only considering the classical limit of Quantum Statistical Mechanics.
[5] It is important, with this regard, to stress that the Second Principle assumes the status of an exact law only when the thermodynamical limit $N \rightarrow +\infty$ is performed and hence *large deviations* become impossible with certainty.
[6] It is with this regard extremely significant that the relevance of the non-probabilistic algorithmic approach to Information Theory was discovered by the same Andrei Nikolaevic Kolmogorov [16] who is also the father of the nowadays generally accepted axiomatic measure-theoretic foundation of (Classical) Probability Theory [17].
[7] Such a conviction is strengthened taking into account, for a moment, also General Relativity and incorporating Black-holes' Thermodynamics (see the 12^{th} chapter "Black holes" of [26]) in the analysis.

In the framework of Nonrelativistic Quantum Mechanics the situation is even more delicate.
According to the orthodox Copenhagen interpretation there exist in it two different dynamical processes:

1. the unitary evolution governing the dynamics of any *closed* system (that following Roger Penrose's terminology, see for instance [27], we will denote as the *U-process*) and ruled by the Schrödinger's equation.

2. the reduction process (that following again Penrose's terminology we will call the *R-process*) occurring when a measurement is performed on a quantum system.

While no doubt exists as to the time-reversal symmetry of the *U-process*, a great debate exists as to the time-reversal symmetry of the *R-process*.

If the *R-process* occurring when a system S is subjected to a measurement could be derived as a particular case of the *U-process* for the system $S + D$ (where D is the experimental device involved in such a measurement) then one could immediately infer that the *R-process* has time-reversal symmetry too.

The impossibility of such a task (first attempted by John Von Neumann in the 6^{th} chapter "The Measuring Process" of [28]) constitutes the so called *Measurement Problem of Quantum Mechanics*.

One of the more popular proposed strategies to circumvent such a problem consists in assuming that during the measurement the composite system S+D is weakly coupled to the environment E and is hence a *weakly open* system (defined as an *open* system for which the *weak coupling limit*, for whose mathematical definition we demand to the 7^{th} chapter "Open Systems" of [37], to the 3^{th} chapter "Evolution of an Open System" of [38] or to the 4^{th} chapter "Reversibility, Irreversibility and Macroscopic Causality" of [39], holds).

Since the fact that the dynamics of a *closed* system is ruled by the *U-process* implies that the dynamics of a *weakly open* system is described (in the Heisenberg picture) by a *completely positive unity-preserving* dynamical semigroup, it is then shown that such a dynamics induces a sort of *superselection rule* (see [29], [30], [31], [32]) that cancels any superposition between different eigenstates of the device operator furnishing the result of the measurement.

Taking into account the Fundamental Theorem of Noncommutative Probability [8] one can describe the quantum dynamics of *weakly open* systems (i.e. the Theory of Dynamical Semigroups) in the mathematical very elegant way available in the mentioned literature.

Anyway nor such a mathematical restyling neither the fact of taking into account also non-projective measurements allows to bypass the fact that, using the words of John Bell [40], such an explanation (called the *decoherence*'s solution or the *environment induced superselection rules'* solution) works *"for all practical purposes"* (locution usually shortened as *FAPP*) but is inconsistent from a logical point of view.

In fact:

1. modern technology allows to manage single particles (for instance a single photon may be obtained by attenuating the coherent output $|z>:= \exp(-\frac{|z|^2}{2})\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n>$ of a laser [9] so that $|z| < r_{threshold}$ or by some other more efficient photodetector; see the section 7.4 "Optical photon quantum computer" of [41]; for the related intriguing issue concerning the limit existing in acquiring information about the wave-function of a single system see [42]; let us stress, with this regard that the impossibility of the determining the wave function of a single quantum system cannot be used to infer that such a concept is not mathematically and physically well-defined), with the consequence that the trick of claiming that the state of a finite quantum system corresponds only to an ensemble of particles and is hence specified completely by a density operator (and not by a ray) on an Hilbert space is trivially experimentally falsified; all the more so such a falsification applies to interpretations according to which the state of a finite system is described by a ray on an Hilbert space, but corresponds only to an ensemble of particles [10].

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[8] stating that the *category* having as objects the classical probability spaces and as *morphisms* the *endomorphisms* (*automorphisms*) of such spaces is equivalent to the category having as objects the *abelian algebraic probability spaces* and as morphisms the endomorphisms (*automorphisms*) of such spaces; see [33], [34], [35], [36].

[9] where of course $|n>$ is an eigentstate of the number operator $n := a^{\dagger} a$, with $[a, a^{\dagger}] = 1$, of a fixed mode of the electromagnetic field.

[10] Owing to the mentioned Fundamental Theorem of Noncommutative Probability, the statistical structure of a quantum system is indeed specified by a noncommutative probability space, i.e. a couple (A, ω) where A is a noncommutative $W^{\star}$-algebra while ω is a state on it (i.e. $\omega \in \mathcal{S}(A)$) and hence it is ruled by a statistical structure different from the classical one. This fact, contrary to what it is sometimes claimed, doesn't anyway solve the Measurement Problem of Quantum Mechanics as we will now briefly show. Let us observe first of all that the assumption that such a quantum system is finite implies that the factor decomposition of A contains only factors of discrete type (i.e. of type I_n $n \in \mathbb{N}_+ \cup \{+\infty\}$), that there exists a separable Hilbert space $\mathcal{H}$ such that A is nothing but the algebra of all the bounded operators over $\mathcal{H}$ (i.e. $A = \mathcal{B}(\mathcal{H})$), and that the state ω is normal and hence there exists a density operator ρ such that $\omega(a) = Tr(\rho a)$ $\forall a \in A$. In the Heisenberg's picture of the motion the U-process is described by a strongly continuous group of inner automorphisms of A. Let us now observe that in the particular case in which the system is a single particle the removal of any

2. there is no reason to assume that, as a matter of principle, it is impossible to isolate the system S+D and hence to treat it as a closed system.

3. such an explanation simply translates the problem of obtaining the *R-process* for S as an *U-process* for S+D into the problem of obtaining it from the *U-process* for the composite system S + D + E [11].

Decoherence is actually a dramatically true phenomenon responsible for the appearance of a macroscopical classical behavior from an underlying microscopic Quantum Theory (and a hated enemy fought through quantum error-correcting codes with success only for a few qubits by those involved in the physical implementation of quantum-computation; see for instance the 10^{th} chapter "Quantum error-correction" of [41]) but is not, in our modest opinion, a conceptually consistent solution to the *Measurement Problem of Quantum Mechanics*.

What is relevant to our present purposes is anyway simply to stress that, whichever viewpoint is assumed as to the interpretational problems of Quantum Mechanics and despite the opposite claim so often uncritically assumed in the literature, there are reasons to suppose the also the *R-process* may occur in a time-reversal invariant way (see for instance [46]), with the net effect that the conflict between the microscopic reversibility and the macroscopic thermodynamical irreversibility appears again, with the considerations previously exposed as to Classical Statistical Mechanics easily extendible to the quantum case.

The reasons to assume that probabilistic considerations are not sufficient to explain the existence of the arrow of time are, in that context, even more striking, going from the interplay between Lorentzian Geometry and Thermodynamics showed by the appearance of Kubo-Martin-Schwinger thermal states in Quantum Field Theories on spacetimes presenting a bifurcate Killing horizon [47] to arguments of Quantum Cosmology going far beyond our competence (see for instance [48], [49], [50], the 5^{th} chapter "The Time Arrow of spacetime" and the 6^{th} chapter "The Time Arrow in Quantum Cosmology" of [51]).

After the presented synoptic view about the time-reversal foundational issues let us at last present the content, strongly more modest, of this paper:

to discuss some time-reversal properties in the coupling of quantum angular momenta that might contribute to shed some little light on them.

More specifically, the menu of this paper is the following:

- in the section II we briefly review some basic facts concerning the Coupling Theory and the Recoupling Theory of quantum angular momenta.

- in the section III we show that that the *double time-reversal superselection rule* of Nonrelativistic Quantum Mechanics is redundant.

- in the section IV we analyze which, among the symmetries of the Clebsch-Gordan coefficients, may be inferred through time-reversal considerations.

- in the section V we show how Coupling Theory allows to improve our comprehension of the fact that, in presence of hidden symmetries, Wigner's Theorem concerning Kramers degeneration cannot be applied, by analyzing a set of Z uncoupled massive particles, both in the case in which they are bosons of spin zero and in the case in which they are fermions of spin $\frac{1}{2}$, subjected to a Keplerian force's field and considering the coupling of the 2Z quantum angular momenta resulting by the hidden SO(4) symmetry pertaining to the fact that, apart from the rotational SO(3) symmetry, the Laplace-Runge-Lenz vector is conserved.

We would like to remark that, though not being a paper about Quantum Gravity, this paper could be of some interest for those engaged in such a research field owing to the astonishing link existing between General Relativity and the Theory of Quantum Angular Momentum discovered by Giorgio Ponzano and Tullio Regge in 1968 (i.e that for large values of the quantum number j the Wigner 6j symbol approximates the Einstein-Hilbert action; see [52], the 9^{th} topic "Physical Interpretation, and Asymptotic (Classical) Limits of the Angular Momentum Functions" of

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epistemic ignorance about it implies that the system must be described by in a extremal state $\omega \in \Xi(A)$, i.e. by a state corresponding to a pure density matrix $\rho = |\psi >< \psi|$ (for a suitable $|\psi > \in \mathcal{H}$) and the Measurement Problem appears again in its usual form.

[11] The same can be said as to the iteration of such a procedure in which one introduces a sequence of devices $\{D_n\}_{n \in \mathbb{N}_+}$ and, given $n \in \mathbb{N}_+$, one attempts to derive the *R-process* for $S + D_1, \cdots + D_{n-1}$ as an U-process for $S + D_1, \cdots + D_n$. At every finite $n \in \mathbb{N}_+$ level of this chain, called *the Von Neumann's chain*, one faces a conceptual problem identical to the original one. If contrary one attempts to perform the limit $n \to +\infty$ the emerging peculiarities of the quantum theory of infinite systems (such as the existence of unitarily inequivalent representations of the observables' algebra; see for instance [43], [44], [45]) may contribute to hide the problem under a mountain of mathematical sophistication but not to solve it.

the 5^{th} chapter "Special Topics" of [53] and the section 7.4.3 "Construction in terms of 6j symbols" of [54]) and its deduction in the framework of Loop Quantum Gravity where the physical states of the quantum gravitational field are spin-networks (see the 6^{th} chapter "Quantum space", the 9^{th} chapter "Quantum spacetime: spinfoams" and the appendix A "Groups and recoupling theory" of [6]).

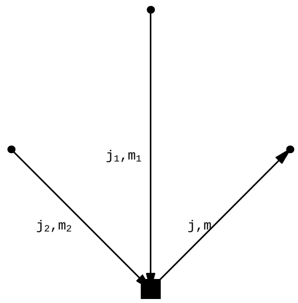

FIG. 1: The Clebsch-Gordan diagram

II. COUPLING AND RECOUPLING OF QUANTUM ANGULAR MOMENTA

Let us recall [55], [56], [57] that the coupling of two quantum angular momenta $\vec{J_1}, \vec{J_2}$ may be represented graphically by a labelled diagram with one vertex (corresponding to the Clebsch-Gordan coefficient $C^{jm}_{j_1 m_1 j_2 m_2}$) having two entering edges (one corresponding to the couple of quantum numbers (j_1, m_1) and the other corresponding to the couple of quantum numbers (j_2, m_2)) and one exiting edge (corresponding to the couple of quantum numbers (j, m)) as represented in the figure 1.

We will call such a graph the Clebsch-Gordan diagram.

Using the new version of the standard package *Combinatorica* for *Mathematica 5* [58] described in [59], the Clebsch-Gordan diagram may be trivially implemented by the following Mathematica expression `cgdiagram[x,y]` where `x` and `y` have to be the list of the labels of, respectively, the 1^{th} and the 2^{th} coupled angular momentum:

```
<< DiscreteMath`Combinatorica`

cgdiagram[x_, y_] :=
  SetGraphOptions[
    FromOrderedPairs[{{1, 3}, {2, 3}, {3, 4}}], {3,
      VertexStyle -> Box[Large]},
    EdgeLabel -> {"j" <> ToString[x] <> "m" <> ToString[x],
        "j" <> ToString[y] <> "m" <> ToString[y],
        "j" <> ToString[Join[x, y]] <> "m" <> ToString[Join[x, y]]}]
```

Demanding to the mentioned literature for the many existing explicit expressions for the Clebsch-Gordan coefficient and its meaning as the probability amplitude $C^{jm}_{j_1 m_1 j_2 m_2} :=< j_1, j_2, m_1, m_2 | j_1, j_2, j, m >$, the basic thing we will need to know about it is that it is always real and that:

Proposition II.1

$$C^{jm}_{j_1 m_1 j_2 m_2} \neq 0 \;\Rightarrow\; j \in \frac{\mathbb{N}}{2} \cap [|j_1 - j_2|, j_1 + j_2] \;\wedge\; m = m_1 + m_2$$
$$\forall m_1 \in \{-j_1, \cdots, j_1\}, \forall m_2 \in \{-j_2, \cdots, j_2\}, \forall m \in \{-j, \cdots, j\}, \forall j_1, j_2, j \in \frac{\mathbb{N}}{2} \quad (2.1)$$

where, given $a, b \in \mathbb{R} : b - a \in \mathbb{N}$, we have adopted the notation:

$$\{a, \cdots, b\} \;:=\; \{a + k \,,\, k \in \mathbb{N} \cap [0, b - a]\} \quad (2.2)$$

Let us now observe that the proposition II.1 implies that:

Corollary II.1

$$C^{jm}_{j_1 m_1 j_2 m_2} \neq 0 \;\Rightarrow\; j \in \{|j_1 - j_2|, \cdots, j_1 + j_2\}$$
$$\forall m_1 \in \{-j_1, \cdots, j_1\}, \forall m_2 \in \{-j_2, \cdots, j_2\}, \forall m \in \{-j, \cdots, j\}, \forall j_1, j_2, j \in \frac{\mathbb{N}}{2} \quad (2.3)$$

PROOF:

Let us consider the different cases:

1. in the case $j_1, j_2 \in \mathbb{N}$ then obviously $|j_1 - j_2| \in \mathbb{N}$. Since $m_1 \in \{-j_1, \cdots, j_1\}$ and $m_2 \in \{-j_2, \cdots, j_2\}$ it follows that $m_1, m_2 \in \mathbb{N}$ and hence, obviously, $m_1 + m_2 \in \mathbb{N}$.

   Let us consider a $j \in \frac{\mathbb{N}}{2} \cap [|j_1 - j_2|, j_1 + j_2]$ such that $j - |j_1 - j_2| \notin \mathbb{N}$ that is, following the notation of section A, $j \in \mathbb{H}$. Since $m \in \{-j, \cdots, j\}$, the proposition A.1 implies that $m \in \mathbb{H}$ with the consequence that $m \neq m_1 + m_2$ and hence $C^{jm}_{j_1 m_1 j_2 m_2} = 0$.

2. in the case $j_1, j_2 \in \mathbb{H}$ then, by the proposition A.1 $|j_1 - j_2| \in \mathbb{N}$. Since $m_1 \in \{-j_1, \cdots, j_1\}$ and $m_2 \in \{-j_2, \cdots, j_2\}$ it follows that $m_1, m_2 \in \mathbb{N}$ and hence, obviously, $m_1 + m_2 \in \mathbb{N}$.

   Let us consider a $j \in \frac{\mathbb{N}}{2} \cap [|j_1 - j_2|, j_1 + j_2]$ such that $j - |j_1 - j_2| \notin \mathbb{N}$ that is $j \in \mathbb{H}$. Since $m \in \{-j, \cdots, j\}$, the proposition A.1 implies that $m \in \mathbb{H}$ with the consequence that $m \neq m_1 + m_2$ and hence $C^{jm}_{j_1 m_1 j_2 m_2} = 0$.

3. in the case $j_1 \in \mathbb{N}$ while $j_2 \in \mathbb{H}$ then, by the proposition A.1 $|j_1 - j_2| \in \mathbb{H}$. Since $m_1 \in \{-j_1, \cdots, j_1\}$ and $m_2 \in \{-j_2, \cdots, j_2\}$, the proposition A.1 implies that $m_1 \in \mathbb{N}$, that $m_2 \in \mathbb{H}$ and hence that $m_1 + m_2 \in \mathbb{H}$.

   Let us consider a $j \in \frac{\mathbb{N}}{2} \cap [|j_1 - j_2|, j_1 + j_2]$ such that $j - |j_1 - j_2| \notin \mathbb{N}$ that is $j \in \mathbb{N}$. Since $m \in \{-j, \cdots, j\}$, the proposition A.1 implies that $m \in \mathbb{N}$ with the consequence that $m \neq m_1 + m_2$ and hence $C^{jm}_{j_1 m_1 j_2 m_2} = 0$.

■

Furthermore, since $|C^{jm}_{j_1 m_1 j_2 m_2}|^2$ is the probability that a measurement of $\vec{J}^2 := J_1^2 + J_2^2 + J_3^2$ and J_3 when the system is in the state $|j_1, j_2, m_1, m_2 >$ give, respectively, the value $j(j+1)$ and the value m, the normalization of probabilities, combined with the application of the proposition II.1 and the corollary II.1, implies that:

Proposition II.2

$$\sum_{j=|j_1-j_2|}^{j_1+j_2} \sum_{m=-j}^{j} |C^{jm}_{j_1 m_1 j_2 m_2}|^2 = 1 \tag{2.4}$$

Let us now suppose to have $n \in \mathbb{N} : n \geq 2$ quantum angular momenta $\vec{J}_1, \cdots, \vec{J}_n$.
Introduced the operators:

$$\vec{J} := \sum_{i=1}^{n} \vec{J}_i \tag{2.5}$$

$$\vec{J}_{(1 \cdots k)} := \sum_{i=1}^{k} \vec{J}_i \quad k \in \{2, \cdots, n-1\} \tag{2.6}$$

we have that each of the possible $(2n-3)!!$ [12] *coupling schemes* may be represented by a labelled oriented tree with n entering edges (corresponding to the n couples of quantum numbers $(j_1, m_1), \cdots, (j_n, m_n))$ and one exiting edge (corresponding to (j,m)) with (n-1) vertices, each with 2 entering edges and one exiting edge; clearly all these labelled oriented trees may be obtained gluing together in all the possible ways n-1 Clebsch-Gordan diagrams, operation that, in the Mathematica 5 setting described above, may be easily implemented through suitable combinations of the Mathematica expressions `Contraction`, `AddEdges` and `DeleteEdges` combined with the following Mathematica expressions useful to manage the *symmetric group of order n* S_n, i.e. the group of all the permutations of n objects:

---

[12] Let us recall that:

$$n!! := \begin{cases} \prod_{\{k \in \mathbb{E} : k \leq n\}} k, & \text{if } n \in \mathbb{E}; \\ \prod_{\{k \in \mathbb{O} : k \leq n\}} k, & \text{f } n \in \mathbb{O}. \end{cases} \tag{2.7}$$

has not to be confused with:

$$(n!)! = \prod_{k=1}^{n!} k = \prod_{k=1}^{\prod_{l=1}^{n} l} k \tag{2.8}$$

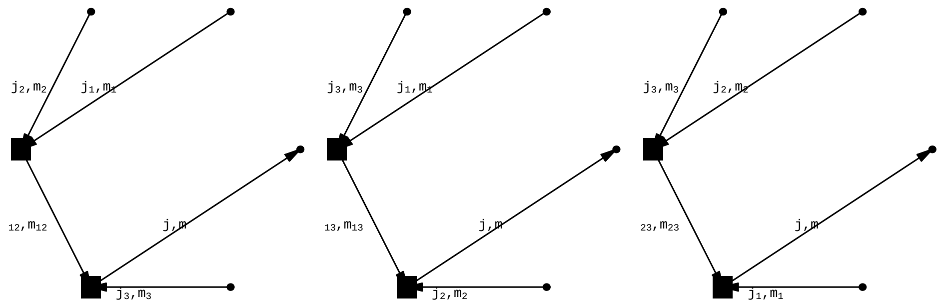

FIG. 2: The diagrams corresponding to the three *coupling schemes* of three angular momenta

```
symmetrize[f_,listofarguments_]:=(1/Factorial[Length[listofarguments]])*
    Sum[Apply[f,
        Permute[listofarguments,
          UnrankPermutation[p,Length[listofarguments]]]],{p,1,
        Factorial[Length[listofarguments]]}]

antisymmetrize[f_,listofarguments_]:=(1/Factorial[Length[listofarguments]])*
    Sum[SignaturePermutation[UnrankPermutation[p,Length[listofarguments]]]*
        Apply[f,Permute[listofarguments,
            UnrankPermutation[p,Length[listofarguments]]]],{p,1,
        Factorial[Length[listofarguments]]}]

exchange[listofarguments_,couple_]:=
  Permute[listofarguments,
    FromCycles[
      Join[{{couple[[1]],couple[[2]]}},
        Table[{Complement[Range[Length[listofarguments]],couple][[i]]},{i,1,
            Length[listofarguments]-2}]]]]

operatorofexchange[f_,listofarguments_,couple_]:=
  Apply[f,exchange[listofarguments,couple]]
```

For instance the three *coupling schemes* existing in the coupling of three angular momenta are represented in the figure 2.

The complete orthonormal bases corresponding to different *coupling schemes* are related by suitable unitary operators whose information is completely encoded in the *Wigner (3n)-j symbol*. The operation of passing from one *coupling scheme* to an other is called recoupling and the associated theory is called Recoupling Theory, a theory whose representability in diagrammatic way is deeply linked to Knot Theory (see the appendix A "Groups and recoupling theory" of [6], [60] [61] and the section 2.13 "Supplement on Combinatorial Foundations" of [62] for any further information).

In the following we will consider only one of these *coupling schemes*, namely the one in which the $\{\vec{J}_A\}_{A=1}^{n}$ are coupled sequentially:

$$\vec{J}_1 + \vec{J}_2 \,=\, \vec{J}_{12} \;,\; \vec{J}_{12} + \vec{J}_3 \,=\, \vec{J}_{123} \;,\; \cdots \;,\vec{J}_{1\cdots n-1} + \vec{J}_n \,=\, \vec{J} \tag{2.9}$$

Given $j_1, \cdots, j_n \in \frac{\mathbb{N}}{2}$ the eigenvectors corresponding to the two complete set of commuting observables $\{\vec{J}_1, J_{1z}, \vec{J}_2, J_{2z}, \cdots, \vec{J}_n, J_{nz}\}$ and $\{\vec{J}_1, \vec{J}_2, \vec{J}_{12}, \vec{J}_3, \vec{J}_{123}, \cdots, \vec{J}_{n-1}, \vec{J}_{1\cdots n-1}, \vec{J}, J_z\}$ are linked by:

Proposition II.3

$$|j_1, j_2, j_{12}, j_3, j_{123}, \cdots, j_{1\cdots n-1}, j_n, j, m> =$$
$$\sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \cdots \sum_{m_n=-j_n}^{j_n} C^{jm}_{j_{1\cdots n-1} m_{1\cdots n-1} j_n m_n} \cdots C^{j_{123} m_{123}}_{j_{12} m_{12} j_3 m_3} C^{j_{12} m_{12}}_{j_1 m_1 j_2 m_2} |j_1, m_1, \cdots, j_n, m_n>$$
$$\forall m \in \{-j, \cdots, j\}, \forall j \in \{|j_{1\cdots n-1} - j_n|, \cdots, j_{1\cdots n-1} + j_n\}$$
$$\forall j_{1\cdots n-1} \in \{|j_{1\cdots n-2} - j_{n-1}|, \cdots, j_{1\cdots n-2} + j_{n-1}\}, \cdots, \forall j_{123} \in \{|j_{12} - j_3|, \cdots, j_{12} + j_3\}, \forall j_{12} \in \{|j_1 - j_2|, \cdots, j_1 + j_2\} \tag{2.10}$$

where we have adopted (as we will implicitly do from now on) the notation:

$$m_{12} := m_1 + m_2 \tag{2.11}$$

$$m_{1\cdots k} := m_{1\cdots k-1} + m_k = \sum_{i=1}^{k} m_i \ \ \forall k \in \{3, \cdots, n\} \tag{2.12}$$

It may be useful to introduce the following:

Definition II.1

generalized coupling coefficient:

$$\mathcal{C}^{j_1, j_2, j_{12}, j_3, j_{123}, \cdots, j_{1\cdots n-1}, j_n, j, m}_{j_1, m_1, \cdots, j_n, m_n} :=$$
$$< j_1, m_1, \cdots, j_n, m_n | j_1, j_2, j_{12}, j_3, j_{123}, \cdots, j_{1\cdots n-1}, j_n, j, m > = C^{jm}_{j_{1\cdots n-1} m_{1\cdots n-1} j_n m_n} \cdots C^{j_{123} m_{123}}_{j_{12} m_{12} j_3 m_3} C^{j_{12} m_{12}}_{j_1 m_1 j_2 m_2} \tag{2.13}$$

implemented computationally by the following Mathematica expression:

```
generalizedcouplingcoefficient[listofpedices_, listofapices_] :=
  Product[ ClebschGordan[{listofapices[[2*i - 1]] ,
        Sum[listofpedices[[2*j]], {j, 1, i}] }, {listofapices[[2*i]],
        listofpedices[[2 + 2*i]]}, {listofapices[[2*i + 1]],
        If[i == Length[listofpedices]/2 - 1, Last[ listofapices ]  ,
          Sum[listofpedices[[2*j]], {j, 1, i + 1}] ]}], {i, 1,
      Length[listofpedices]/2 - 1}]
```

Let us observe that clearly the quantum number j can take the following values:

$$j \in \{\mathcal{J}_{min}(j_1, \cdots, j_n), \cdots, \mathcal{J}_{max}(j_1, \cdots, j_n)\} \tag{2.14}$$

where the map $\mathcal{J}_{max} : (\frac{\mathbb{N}}{2})^n \mapsto \frac{\mathbb{N}}{2}$ is defined simply as:

$$\mathcal{J}_{max}(j_1, \cdots, j_n) := \sum_{i=1}^{n} j_i \tag{2.15}$$

while the map $\mathcal{J}_{min} : (\frac{\mathbb{N}}{2})^n \mapsto \frac{\mathbb{N}}{2}$ is defined recursively as:

$$\mathcal{J}_{min}(j_1, \cdots, j_n) := \begin{cases} |j_1 - j_2|, & \text{if } n = 2; \\ \min\{|i - j_n| \,,\, i \in \{\mathcal{J}_{min}(j_1, \cdots, j_{n-1}), \cdots, \mathcal{J}_{max}(j_1, \cdots, j_{n-1})\}\}, & \text{otherwise.} \end{cases} \tag{2.16}$$

Both these maps may be computed through the following Mathematica 5 expressions `jmin[x]` and `jmax[x]`, where `x` is the list $\{j_1, \cdots, j_n\}$:

```
jmaximum[x_,n_]:=Sum[x[[i]],{i,1,n}]

jminimum[x_,n_]:=
  If[n==2, Abs[x[[1]]-x[[2]]],
    Min[Table[Abs[i-x[[n]]] ,{i,jminimum[x,n-1],jmaximum[x,n-1]}]]]

jmin[x_]:=jminimum[x,Length[x]]

jmax[x_]:=jmaximum[x, Length[x]]
```

Taking into account all these considerations together with the proposition II.1 it follows that:

Proposition II.4

$$\mathcal{C}^{j_1,j_2,j_{12},j_3,j_{123},\cdots,j_{1\cdots n-1},j_n,j,m}_{j_1,m_1,\cdots,j_n,m_n} \neq 0 \;\Rightarrow\; j \in \{\mathcal{J}_{min}(j_1,\cdots,j_n),\cdots,\mathcal{J}_{max}(j_1,\cdots,j_n)\} \wedge m = \sum_{i=1}^{n} m_i \tag{2.17}$$

Furthermore the definition II.1 and the proposition II.2 imply that:

Proposition II.5

$$\sum_{j_{12}=|j_1-j_2|}^{j_1+j_2} \sum_{j_{123}=|j_{12}-j_3|}^{j_{12}+j_3} \cdots \sum_{j=|j_{1\cdots n-1}-j_n|}^{j_{1\cdots n-1}+j_n} \sum_{m=-j}^{j} |\mathcal{C}^{j_1,j_2,j_{12},j_3,j_{123},\cdots,j_{1\cdots n-1},j_n,j,m}_{j_1,m_1,\cdots,j_n,m_n}|^2 \;=\; 1 \tag{2.18}$$

III. CONSIDERATIONS ABOUT THE *DOUBLE TIME-REVERSAL SUPERSELECTION RULE* IN NONRELATIVISTIC QUANTUM MECHANICS

The notion of *superselection rule* introduced by Gian Carlo Wick , Arthur Wightman and Eugene Wigner in their celebrated paper [29] is nowadays a corner stone of our comprehension of the mathematical structure of (Special)-Relativistic Quantum Mechanics [63], [64], [65] (i.e. of Quantum Field Theory over the Minkowskian spacetime $(\mathbb{R}^4, \eta)$ [13]).

Since Nonrelativistic Quantum Mechanics may be obtained from Quantum Field Theory over the Minkowski spacetime by taking the limit $c \rightarrow +\infty$, implemented mathematically as the contraction [66] of the Lie algebra $L[\mathcal{P}]$ of the isometries's group of the Minkowski spacetime [14], it would seem reasonable to suppose that also the status of superselection rules in Nonrelativistic Quantum Mechanics should be a settled issue.

Curiously the status of the *double time-reversal superselection rule* is Nonrelativistic Quantum Mechanics is not yet clear.

The *double time-reversal superselection rule* states that the square T^2 of the (antilinear, antiunitary) *time-reversal operator* T (about which see for instance the 26^{th} chapter "Time Inversion" of [67], the Topic 1 "Fundamental Symmmetry Considerations" of the 5^{th} chapter "Special Topics" of [53], the 15^{th} chapter "Invariance and Conservation Theorems, Time Reversal" of [68], the 4^{th} chapter "Symmetries in Quantum Mechanics" of [69], the 13^{th} chapter "Discrete Symmetries" of [70], the 10^{th} chapter "Time Reversal" of [71] and, last but not least, [13]) is a *superselection charge*.

Let us define as *even under double time-reversal* any state $|\psi >$ such that $T^2|\psi >= |\psi >$.

Similarly, let us define as *odd under double time-reversal* any state $|\psi >$ such that $T^2|\psi >= -|\psi >$.

The physical argument proposed by Wick, Wightman and Wigner supporting the existence of the *double time-reversal superselection rule* is the following:

since inverting twice the arrow of time should have no physical effect, given a state $|E>$ even under double time-reversal and a state $|O>$ odd under double time-reversal, one should have that the states $|E> +|O>$ and the state $T^2(|E> +|O>)$ should have the same physical content.

But since obviously:

$$T^2(|E> +|O>) \ = \ T^2|E> +T^2|O> = \ |E> -|O> \tag{3.9}$$

this is not possible, and hence superposition of states with different parity under double time-reversal should be banned.

The status of such an argument is not so obvious (for arguments against it differing by the ones we are going to present we demand to the 11^{th} chapter "Superselection Rules" of [72]) as we will now show.

Let us start, at this purpose, to review the behavior under time-reversal of a generic quantum angular momentum $\vec{J}$:

$$T \vec{J} T^{-1} \ = \ - \vec{J} \tag{3.10}$$

$$T|j,m> = \ i^{2m} |j, -m> \tag{3.11}$$

---

[13] where of course $\eta := \eta_{\mu\nu} dx^{\mu} \otimes dx^{\nu}$, $\eta_{\mu\nu} := diag(-1,1,1,1)$.

[14] i.e. the Lie algebra $L[\mathcal{P}]$ of the Poincaré group $\mathcal{P}$, Lie algebra generated by the Killing vector fields P_{μ} and $M_{\mu\nu}$:

$$[P_{\mu}, P_{\nu}] \ = \ 0 \tag{3.1}$$

$$M_{\nu\mu} \ = \ -M_{\mu\nu} \tag{3.2}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] \ = \ \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\rho} \tag{3.3}$$

$$[M_{\mu\nu}, P_{\sigma}] \ = \ \eta_{\mu\sigma} P_{\nu} - \eta_{\nu\sigma} P_{\mu} \tag{3.4}$$

whose contraction results in the Lie algebra of the Galilei group admitting the basis P_{μ}, K_i such that:

$$M_{0i} \ =: \ c K_i \tag{3.5}$$

$$[P_{\mu}, P_{\nu}] \ = \ 0 \tag{3.6}$$

$$[K_i, K_j] \ = \ \lim_{c \rightarrow +\infty} \frac{1}{c^2} [M_{0i}, M_{0j}] \ = \ 0 \tag{3.7}$$

$$[K_i, P_{\mu}] \ = \ \lim_{c \rightarrow +\infty} \frac{1}{c} [M_{0i}, P_{\mu}] \ = \ 0 \tag{3.8}$$

and hence:

$$T^{2}|j,m> = \; Ti^{2m}|j,-m> = \; i^{-2m}T|j,-m> = \; i^{-4m}|j,m> = \; (-1)^{2j}|j,m> \tag{3.12}$$

where we have used the fact that:

$$i^{-4m} \;=\; (-1)^{2j} \;\; \forall m \in \{-j,\cdots,j\}\,,\, \forall j \in \frac{\mathbb{N}}{2} \tag{3.13}$$

Remark III.1

Let us remark that obviously:

$$(-1)^{2j} \;=\; \begin{cases} +1, & \text{if } j \in \mathbb{N}; \\ -1, & \text{if } j \in \mathbb{H}. \end{cases} \tag{3.14}$$

So the *double time reversal superselection rule* implies the *univalence superselection rule*, defined as the superselection rule having *superselection charge* $(-1)^{2j}$ and hence banning the superposition of states with $j \in \mathbb{N}$ and states with $j \in \mathbb{H}$.

Let us now consider the coupling of $n \in \mathbb{N} : n \geq 2$ quantum angular momenta $\vec{J}_1,\cdots,\vec{J}_n$; given $j_1,\cdots,j_n \in \frac{\mathbb{N}}{2}$ let us observe that:

Proposition III.1

$$(-1)^{2j} \;=\; \begin{cases} +1, & \text{if } |\{j_i \in \mathbb{H}\,,\, i \in \{1,\cdots,n\}\}| \in \mathbb{E}; \\ -1, & \text{if } |\{j_i \in \mathbb{H}\,,\, i \in \{1,\cdots,n\}\}| \in \mathbb{O} \end{cases} \quad \forall j \in \{\mathcal{J}_{min}(j_1,\cdots,j_n),\cdots,\mathcal{J}_{max}(j_1,\cdots,j_n)\} \tag{3.15}$$

PROOF:

The thesis follows combining the equation 3.14 and the proposition A.1. ■

Proposition III.1 plays an unexpected rule as to time-reversal properties.

Let us observe, first of all, that:

$$T|j_1,m_1,\cdots,j_n,m_n> = \; i^{2\sum_{i=1}^{n} m_i}|j_1,-m_1,\cdots,j_n,-m_n> \tag{3.16}$$

$$T|j_1,j_2,j_{12},j_3,j_{123},\cdots,j_{1\cdots n-1},j_n,j,m> = \; i^{2m}|j_1,j_2,j_{12},j_3,j_{123},\cdots,j_{1\cdots n-1},j_n,j,-m> \tag{3.17}$$

Applying, from the other side, the time-reversal operator to the proposition II.3 it follows that:

$$T|j_1,j_2,j_{12},j_3,j_{123},\cdots,j_{1\cdots n-1},j_n,j,m> = \\ T\sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2}\cdots\sum_{m_n=-j_n}^{j_n} C^{jm}_{j_{1\cdots n-1}m_{1\cdots n-1}j_n m_n}\cdots C^{j_{123}m_{123}}_{j_{12}m_{12}j_3 m_3}C^{j_{12}m_{12}}_{j_1 m_1 j_2 m_2}|j_1,m_1,\cdots,j_n,m_n> \tag{3.18}$$

$$T|j_1,j_2,j_{12},j_3,j_{123},\cdots,j_{1\cdots n-1},j_n,j,m> = \\ \sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2}\sum_{m_3=-j_3}^{j_3}\cdots\sum_{m_n=-j_n}^{j_n} TC^{jm}_{j_{1\cdots n-1}m_{1\cdots n-1}j_n m_n}\cdots C^{j_{123}m_{123}}_{j_{12}m_{12}j_3 m_3}C^{j_{12}m_{12}}_{j_1 m_1 j_2 m_2}|j_1,m_1,\cdots,j_n,m_n> \tag{3.19}$$

$$T|j_1,j_2,j_{12},j_3,j_{123},\cdots,j_{1\cdots n-1},j_n,j,m> = \\ \sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2}\sum_{m_3=-j_3}^{j_3}\cdots\sum_{m_n=-j_n}^{j_n} \overline{C^{jm}_{j_{1\cdots n-1}m_{1\cdots n-1}j_n m_n}\cdots C^{j_{123}m_{123}}_{j_{12}m_{12}j_3 m_3}C^{j_{12}m_{12}}_{j_1 m_1 j_2 m_2}}T|j_1,m_1,\cdots,j_n,m_n> \tag{3.20}$$

where we have used the anti-linearity of T. Hence:

$$T|j_1, j_2, j_{12}, j_3, j_{123}, \cdots, j_{1\cdots n-1}, j_n, j, m> = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \sum_{m_3=-j_3}^{j_3} \cdots \sum_{m_n=-j_n}^{j_n} C^{jm}_{j_{1\cdots n-1} m_{1\cdots n-1} j_n m_n} \cdots C^{j_{123} m_{123}}_{j_{12} m_{12} j_3 m_3} C^{j_{12} m_{12}}_{j_1 m_1 j_2 m_2} i^{2\sum_{i=1}^n m_i} |j_1, -m_1, \cdots, j_n, -m_n> \tag{3.21}$$

where in the last passage we have used the equation 3.16 and the fact that the Clebsch-Gordan coefficients are reals.
Comparing the equation 3.17 and the equation 3.21 it follows that:

$$\begin{gathered} |j_1, j_2, j_{12}, j_3, j_{123}, \cdots, j_{1\cdots n-1}, j_n, j, -m> = \\ \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \sum_{m_3=-j_3}^{j_3} \cdots \sum_{m_n=-j_n}^{j_n} C^{jm}_{j_{1\cdots n-1} m_{1\cdots n-1} j_n m_n} \cdots C^{j_{123} m_{123}}_{j_{12} m_{12} j_3 m_3} C^{j_{12} m_{12}}_{j_1 m_1 j_2 m_2} i^{2(\sum_{i=1}^n m_i - m)} |j_1, -m_1, \cdots, j_n, -m_n> \\ \forall m \in \{-j, \cdots, j\}, \forall j \in \{|j_{1\cdots n-1} - j_n|, \cdots, j_{1\cdots n-1} + j_n\} \\ \forall j_{1\cdots n-1} \in \{|j_{1\cdots n-2} - j_{n-1}|, \cdots, j_{1\cdots n-2} + j_{n-1}\}, \cdots, \forall j_{123} \in \{|j_{12} - j_3|, \cdots, j_{12} + j_3\}, \forall j_{12} \in \{|j_1 - j_2|, \cdots, j_1 + j_2\} \end{gathered} \tag{3.22}$$

that using the proposition II.1 reduces to:

Proposition III.2

$$\begin{gathered} |j_1, j_2, j_{12}, j_3, j_{123}, \cdots, j_{1\cdots n-1}, j_n, j, -m> = \\ \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \sum_{m_3=-j_3}^{j_3} \cdots \sum_{m_n=-j_n}^{j_n} C^{jm}_{j_{1\cdots n-1} m_{1\cdots n-1} j_n m_n} \cdots C^{j_{123} m_{123}}_{j_{12} m_{12} j_3 m_3} C^{j_{12} m_{12}}_{j_1 m_1 j_2 m_2} |j_1, -m_1, \cdots, j_n, -m_n> \\ \forall m \in \{-j, \cdots, j\}, \forall j \in \{|j_{1\cdots n-1} - j_n|, \cdots, j_{1\cdots n-1} + j_n\} \\ \forall j_{1\cdots n-1} \in \{|j_{1\cdots n-2} - j_{n-1}|, \cdots, j_{1\cdots n-2} + j_{n-1}\}, \cdots, \forall j_{123} \in \{|j_{12} - j_3|, \cdots, j_{12} + j_3\}, \forall j_{12} \in \{|j_1 - j_2|, \cdots, j_1 + j_2\} \end{gathered} \tag{3.23}$$

Furthermore:

$$T^2 |j_1, m_1, \cdots, j_n, m_n> = (-1)^{2\sum_{i=1}^n j_i} |j_1, m_1, \cdots, j_n, m_n> \tag{3.24}$$

$$T^2 |j_1, j_2, j_{12}, j_3, j_{123}, \cdots, j_{1\cdots n-1}, j_n, j, m> = (-1)^{2j} |j_1, j_2, j_{12}, j_3, j_{123}, \cdots, j_{1\cdots n-1}, j_n, j, m> \tag{3.25}$$

Applying, from the other side, the square of the time-reversal operator to the proposition II.3 it follows that:

$$T^2 |j_1, j_2, j_{12}, j_3, j_{123}, \cdots, j_{1\cdots n-1}, j_n, j, m> = T^2 \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \sum_{m_3=-j_3}^{j_3} \cdots \sum_{m_n=-j_n}^{j_n} C^{jm}_{j_{1\cdots n-1} m_{1\cdots n-1} j_n m_n} \cdots C^{j_{123} m_{123}}_{j_{12} m_{12} j_3 m_3} C^{j_{12} m_{12}}_{j_1 m_1 j_2 m_2} |j_1, m_1, \cdots, j_n m_n> \tag{3.26}$$

$$T^2 |j_1, j_2, j_{12}, j_3, j_{123}, \cdots, j_{1\cdots n-1}, j_n, j, m> = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \sum_{m_3=-j_3}^{j_3} \cdots \sum_{m_n=-j_n}^{j_n} T^2 C^{jm}_{j_{1\cdots n-1} m_{1\cdots n-1} j_n m_n} \cdots C^{j_{123} m_{123}}_{j_{12} m_{12} j_3 m_3} C^{j_{12} m_{12}}_{j_1 m_1 j_2 m_2} |j_1, m_1, \cdots, j_n, m_n> \tag{3.27}$$

$$T^2 |j_1, j_2, j_{12}, j_3, j_{123}, \cdots, j_{1\cdots n-1}, j_n, j, m> = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \sum_{m_3=-j_3}^{j_3} \cdots \sum_{m_n=-j_n}^{j_n} C^{jm}_{j_{1\cdots n-1} m_{1\cdots n-1} j_n m_n} \cdots C^{j_{123} m_{123}}_{j_{12} m_{12} j_3 m_3} C^{j_{12} m_{12}}_{j_1 m_1 j_2 m_2} T^2 |j_1, m_1, \cdots, j_n, m_n> \tag{3.28}$$

where in the last passage we have used the linearity of T^2. Hence:

$$T^2|j_1, j_2, j_{12}, j_3, j_{123}, \cdots, j_{1\cdots n-1}, j_n, j, m> =$$
$$\sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \sum_{m_3=-j_3}^{j_3} \cdots \sum_{m_n=-j_n}^{j_n} C^{jm}_{j_{1\cdots n-1} m_{1\cdots n-1} j_n m_n} \cdots C^{j_{123} m_{123}}_{j_{12} m_{12} j_3 m_3} C^{j_{12} m_{12}}_{j_1 m_1 j_2 m_2} (-1)^{2\sum_{i=1}^{n} j_i} |j_1, m_1, \cdots, j_n, m_n> \tag{3.29}$$

where we have used the equation 3.24.

Comparing the equation 3.25 and the equation 3.29 it follows that:

Proposition III.3

$$|j_1, j_2, j_{12}, j_3, j_{123}, \cdots, j_{1\cdots n-1}, j_n, j, m> =$$
$$\sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \sum_{m_3=-j_3}^{j_3} \cdots \sum_{m_n=-j_n}^{j_n} C^{jm}_{j_{1\cdots n-1} m_{1\cdots n-1} j_n m_n} \cdots C^{j_{123} m_{123}}_{j_{12} m_{12} j_3 m_3} C^{j_{12} m_{12}}_{j_1 m_1 j_2 m_2} (-1)^{2(\sum_{i=1}^{n} j_i - j)} |j_1, m_1, \cdots, j_n, m_n>$$
$$\forall m \in \{-j, \cdots, j\}, \forall j \in \{|j_{1\cdots n-1} - j_n|, \cdots, j_{1\cdots n-1} + j_n\}$$
$$\forall j_{1\cdots n-1} \in \{|j_{1\cdots n-2} - j_{n-1}|, \cdots, j_{1\cdots n-2} + j_{n-1}\}, \cdots, \forall j_{123} \in \{|j_{12} - j_3|, \cdots, j_{12} + j_3\}, \forall j_{12} \in \{|j_1 - j_2|, \cdots, j_1 + j_2\} \tag{3.30}$$

Let us now observe that while no compatibility problem exists between the proposition II.3 and the proposition III.2, the compatibility between the proposition II.3 and the proposition III.3 requires that:

$$(-1)^{2(\sum_{i=1}^{n} j_i - j)} = 1 \ \ \forall j \in \{\mathcal{J}_{min}(j_1, \cdots, j_n), \cdots, \mathcal{J}_{max}(j_1, \cdots, j_n)\}, \forall j_1, \cdots, j_n \in \frac{\mathbb{N}}{2} \tag{3.31}$$

But equation 3.31 is a trivial consequence of the proposition III.1.

Remark III.2

Let us now observe that the angular momentum $\vec{J}$ of a particle is obtained coupling its *orbital angular momentum* $\vec{L}$ and its *spin* $\vec{S}$.

The particle is defined to be a *boson* if its univalence is equal to $+1$ while it is defined to be a *fermion* if its univalence is equal to -1.

Since the *univalence* of the *orbital angular momentum* is always equal to $+1$, the proposition III.1 implies that the univalence of the particle is equal to the univalence of its spin.

Remark III.3

The remark III.2 allows to give the following characterization of bosons and fermions:

1. a *fermion* is a particle containing an odd number of *fermions*
2. a *boson* is a particle containing an even number of *fermions* [15]

that, restricting the analysis to particles composed by a finite number of "basic particles" of known univalence [16] and representing mathematically the predicate *"being a subparticle"* with the set-theoretic membership relation , may be implemented through the following Mathematica expression `fermionQ[x]` (where `x` has to be a list whose atomic elements are the univalences of the "basic particles"):

---

[15] Let us recall that zero is an even number.

[16] Let us remark that we have not assumed that these "basic particles" are elementary; we could choose as *"basic particles"*, for instance, the atom of He_4 (known to be a *boson*), and the atom of He_3 (known to be a *fermion*) arriving to the same conclusion, as to the univalence's attribution, that we would have reached if we had chosen as "basic particles" the proton, the neutron and the electron (all known to be fermions) and we had inferred that the atom of He_3 is a fermion by the fact that it is composed by 1 neutron, 2 protons and 2 electrons and we had inferred that the atom of He_4 is a boson by the fact that it is composed by 2 neutrons, 2 protons and 2 neutrons, or if we had chosen as "basic particles" the electron, the quark up and the quark down (all known to be fermions) and, considering that the proton is composed by 2 quarks up and 1 quark down while the neutron is composed by 1 quark up and 2 quarks down, we had inferred that the atom of He_3 is a fermion by the fact that it is composed by 2 electrons, 5 quarks up and 4 quarks down and we had inferred that the atom of He_4 is a boson by the fact that it is composed by 2 electrons, 6 quarks up and 6 quarks down.

```
fermionQ[x_] := If[AtomQ[x], x == -1, OddQ[Length[Select[x,
fermionQ]]]]
```

Remark III.4

Let us now observe that since the univalence of a compound particle is, as stated by the remark III.3, already determined by the Coupling Theory, the *double time-reversal superselection rule* is redundant.

IV. SYMMETRIES OF THE CLEBSCH-GORDAN COEFFICIENTS DEDUCIBLE FROM TIME-REVERSAL CONSIDERATIONS

In the previous section we have actually used only the anti-linearity of T while we have never used its anti-unitarity. Resorting to this property allows, indeed, to derive some symmetry properties of the Clebsch-Gordan coefficients.

Given a state $|\psi>$ let us introduce the following useful and suggestive (one can look at the overleft arrow as the inverted arrow of time) notation:

$$\overleftarrow{|\psi>} := T|\psi> \tag{4.1}$$

Considering again the setting of the previous section, let us then rewrite the equation 3.16 as:

$$\overleftarrow{|j_1, m_1, \cdots, j_n, m_n>} = i^{2\sum_{i=1}^{n} m_i} |j_1, -m_1, \cdots, j_n, -m_n> \tag{4.2}$$

and let us rewrite the equation 3.17 as:

$$\overleftarrow{|j_1, j_2, j_{12}, j_3, j_{123}, \cdots, j_{1\cdots n-1}, j_n, j, m>} = i^{2m} |j_1, j_2, j_{12}, j_3, j_{123}, \cdots, j_{1\cdots n-1}, j_n, j, -m> \tag{4.3}$$

Hence:

$$\begin{aligned}&\overleftarrow{< j_1, m_1, \cdots, j_n, m_n}\overleftarrow{|j_1, j_2, j_{12}, j_3, j_{123}, \cdots, j_{1\cdots n-1}, j_n, j, m>} = \\ &= i^{2(m-\sum_{i=1}^{n} m_i)} < j_1, -m_1, \cdots, j_n, -m_n|j_1, j_2, j_{12}, j_3, j_{123}, \cdots, j_{1\cdots n-1}, j_n, j, -m>\end{aligned} \tag{4.4}$$

By the anti-unitarity of T it follows that:

$$\begin{aligned}&\overleftarrow{< j_1, m_1, \cdots, j_n, m_n}\overleftarrow{|j_1, j_2, j_{12}, j_3, j_{123}, \cdots, j_{1\cdots n-1}, j_n, j, m>} = \\ &\overline{< j_1, m_1, \cdots, j_n, m_n|j_1, j_2, j_{12}, j_3, j_{123}, \cdots, j_{1\cdots n-1}, j_n, j, m>}\end{aligned} \tag{4.5}$$

and hence:

$$\begin{aligned}&\overline{< j_1, m_1, \cdots, j_n, m_n|j_1, j_2, j_{12}, j_3, j_{123}, \cdots, j_{1\cdots n-1}, j_n, j, m>} = \\ &i^{2(m-\sum_{i=1}^{n} m_i)} < j_1, -m_1, \cdots, j_n, -m_n|j_1, j_2, j_{12}, j_3, j_{123}, \cdots, j_{1\cdots n-1}, j_n, j, -m>\end{aligned} \tag{4.6}$$

Since by the proposition II.3:

$$\begin{aligned}&< j_1, m_1, \cdots, j_n, m_n|j_1, j_2, j_{12}, j_3, j_{123}, \cdots, j_{1\cdots n-1}, j_n, j, m> = \\ &C^{jm}_{j_{1\cdots n-1} m_{1\cdots n-1} j_n m_n} \cdots C^{j_{123} m_{123}}_{j_{12} m_{12} j_3 m_3} C^{j_{12} m_{12}}_{j_1 m_1 j_2 m_2}\end{aligned} \tag{4.7}$$

the equation 4.6 may be rewritten as:

$$\begin{aligned}&\overline{C^{jm}_{j_{1\cdots n-1} m_{1\cdots n-1} j_n m_n} \cdots C^{j_{123} m_{123}}_{j_{12} m_{12} j_3 m_3} C^{j_{12} m_{12}}_{j_1 m_1 j_2 m_2}} = \\ &i^{2(m-\sum_{i=1}^{n} m_i)} C^{j,-m}_{j_{1\cdots n-1},-m_{1\cdots n-1},j_n,-m_n} \cdots C^{j_{123},-m_{123}}_{j_{12},-m_{12},j_3,-m_3} C^{j_{12},-m_{12}}_{j_1,-m_1,j_2,-m_2}\end{aligned} \tag{4.8}$$

(where we have used commas to separate the indices in order to avoid the notational ambiguity resulting by the minus signs) that using the reality of the Clebsch-Gordan coefficients gives:

$$\begin{aligned}&C^{jm}_{j_{1\cdots n-1} m_{1\cdots n-1} j_n m_n} \cdots C^{j_{123} m_{123}}_{j_{12} m_{12} j_3 m_3} C^{j_{12} m_{12}}_{j_1 m_1 j_2 m_2} = \\ &i^{2(m-\sum_{i=1}^{n} m_i)} C^{j,-m}_{j_{1\cdots n-1},-m_{1\cdots n-1},j_n,-m_n} \cdots C^{j_{123},-m_{123}}_{j_{12},-m_{12},j_3,-m_3} C^{j_{12},-m_{12}}_{j_1,-m_1,j_2,-m_2} \\ &\forall m_1 \in \{-j_1, \cdots, j_1\}, \cdots, \forall m_n \in \{-j_n, \cdots, j_n\}, \forall m \in \{-j, \cdots, j\}, \\ &\forall j \in \{|j_{1\cdots n-1} - j_n|, \cdots, j_{1\cdots n-1} + j_n\}, \forall j_{1\cdots n-1} \in \{|j_{1\cdots n-2} - j_{n-1}|, \cdots, j_{1\cdots n-2} + j_{n-1}\}, \cdots, \\ &\forall j_{123} \in \{|j_{12} - j_3|, \cdots, j_{12} + j_3\}, \forall j_{12} \in \{|j_1 - j_2|, \cdots, j_1 + j_2\}, \forall j_1, \cdots, j_n \in \frac{\mathbb{N}}{2}\end{aligned} \tag{4.9}$$

Using the proposition II.1 it follows that:

Proposition IV.1

$$C^{jm}_{j_{1\cdots n-1}m_{1\cdots n-1}j_n m_n}\cdots C^{j_{123}m_{123}}_{j_{12}m_{12}j_3 m_3}C^{j_{12}m_{12}}_{j_1 m_1 j_2 m_2} = C^{j,-m}_{j_{1\cdots n-1},-m_{1\cdots n-1},j_n,-m_n}\cdots C^{j_{123},-m_{123}}_{j_{12},-m_{12},j_3,-m_3}C^{j_{12},-m_{12}}_{j_1,-m_1,j_2,-m_2}$$
$$\forall m_1\in\{-j_1,\cdots,j_1\},\cdots,\forall m_n\in\{-j_n,\cdots,j_n\},\forall m\in\{-j,\cdots,j\},$$
$$\forall j\in\{|j_{1\cdots n-1}-j_n|,\cdots,j_{1\cdots n-1}+j_n\},\forall j_{1\cdots n-1}\in\{|j_{1\cdots n-2}-j_{n-1}|,\cdots,j_{1\cdots n-2}+j_{n-1}\},\cdots,$$
$$\forall j_{123}\in\{|j_{12}-j_3|,\cdots,j_{12}+j_3\},\forall j_{12}\in\{|j_1-j_2|,\cdots,j_1+j_2\},\forall j_1,\cdots,j_n\in\frac{\mathbb{N}}{2} \quad (4.10)$$

Let us now recall the following:

Proposition IV.2

Basic property of states odd under double time reversal:

$$\overleftarrow{\overleftarrow{|\psi>}} = -|\psi> \Rightarrow \overleftarrow{<\psi}|\psi> = 0 \quad (4.11)$$

PROOF:

Since T is antiunitary:

$$\overleftarrow{<\alpha}|\overleftarrow{\beta>} = \overline{<\alpha|\beta>} = <\beta|\alpha> \quad \forall|\alpha>,|\beta> \quad (4.12)$$

Choosing in particular $|\alpha>:=|\psi>$ and $|\beta>:=\overleftarrow{|\psi>}$ the equation 4.12 gives:

$$\overleftarrow{<\psi}|\psi> = \overleftarrow{<\psi}|\overleftarrow{\overleftarrow{\psi>}} \quad (4.13)$$

that using the hypothesis that $|\psi>$ is odd under double time-reversal becomes:

$$\overleftarrow{<\psi}|\psi> = -\overleftarrow{<\psi}|\psi> \quad (4.14)$$

from which the thesis follows. ■

Applying the proposition IV.2 to the state $|j_1,j_2,j_{12},j_3,j_{123},\cdots,j_{1\cdots n-1},j_n,j,m>$ with $j\in\mathbb{H}$ (where, following the notation introduced in the section A, $\mathbb{H}$ is the set of the *half-odd numbers*), we obtain:

$$<j_1,j_2,j_{12},j_3,j_{123},\cdots,j_{1\cdots n-1},j_n,j,m|\overleftarrow{j_1,j_2,j_{12},j_3,j_{123},\cdots,j_{1\cdots n-1},j_n,j,m>} = 0 \quad (4.15)$$

that, using the proposition II.3 and the equation 3.21, gives:

$$\sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2}\cdots\sum_{m_n=-j_n}^{j_n}\overline{C^{jm}_{j_{1\cdots n-1}m_{1\cdots n-1}j_n m_n}\cdots C^{j_{123}m_3}_{j_{12}m_{12}j_3 m_m}C^{j_{12}m_{12}}_{j_1 m_1 j_2 m_2}}<j_1,m_1,\cdots,j_n,m_n|$$
$$\sum_{m'_1=-j_1}^{j_1}\sum_{m'_2=-j_2}^{j_2}\sum_{m'_3=-j_3}^{j_3}\cdots\sum_{m'_n=-j_n}^{j_n}C^{jm'}_{j_{1\cdots n-1}m'_{1\cdots n-1}j_n m'_n}\cdots C^{j_{123}m'_{123}}_{j_{12}m'_{12}j_3 m'_3}C^{j_{12}m'_{12}}_{j_1 m'_1 j_2 m'_2}i^{2\sum_{i=1}^n m'_i}|j_1,-m'_1,\cdots,j_n,-m'_n> = 0$$
$$(4.16)$$

Since:

$$<j_1,m_1,\cdots,j_n,m_n|j_1,-m'_1,\cdots,j_n,-m'_n> = \prod_{i=1}^n\delta_{m_i,-m'_i} \quad (4.17)$$

(and using the reality of the Clebsch-Gordan coefficients) the only non-zero addends in the left hand side of the equation 4.16 are those with $m'_1=-m_1,\cdots,m'_n=-m_n$ for which one obtains:

Proposition IV.3

$$\sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \cdots \sum_{m_n=-j_n}^{j_n}$$

$$i^{-2(\sum_{i=1}^{n} m_i)} C^{jm}_{j_{1\cdots n-1} m_{1\cdots n-1} j_n m_n} \cdots C^{j_{123} m_{123}}_{j_{12} m_{12} j_3 m_3} C^{j_{12} m_{12}}_{j_1 m_1 j_2 m_2} C^{j,-m}_{j_{1\cdots n-1},-m_{1\cdots n-1} j_n,-m_n} \cdots C^{j_{123},-m_{123}}_{j_{12},-m_{12},j_3,-m_3} C^{j_{12},-m_{12}}_{j_1,-m_1,j_2,-m_2} = 0$$
$$\forall m_1 \in \{-j_1, \cdots, j_1\}, \cdots, \forall m_n \in \{-j_n, \cdots, j_n\}, \forall m \in \{-j, \cdots, j\},$$
$$\forall j \in \{|j_{1\cdots n-1} - j_n|, \cdots, j_{1\cdots n-1} + j_n\} : j \in \mathbb{H}, \forall j_{1\cdots n-1} \in \{|j_{1\cdots n-2} - j_{n-1}|, \cdots, j_{1\cdots n-2} + j_{n-1}\}, \cdots,$$
$$\forall j_{123} \in \{|j_{12} - j_3|, \cdots, j_{12} + j_3\}, \forall j_{12} \in \{|j_1 - j_2|, \cdots, j_1 + j_2\}, \forall j_1, \cdots, j_n \in \frac{\mathbb{N}}{2} \quad (4.18)$$

Let us now compare the proposition IV.1 and the proposition IV.3 with the symmetry properties of the Clebsch-Gordan coefficients (discovered by Tullio Regge in 1954 [73]).

Let us introduce at this purpose the *Wigner 3j symbol*:

$$\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} := \frac{(-1)^{j_1 - j_2 + m}}{\sqrt{2j+1}} C^{j,-m}_{j_1,m_1,j_2,m_2} \quad (4.19)$$

in terms of which the *Regge R-symbol*:

$$\left\| \begin{matrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{matrix} \right\| := \begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} \quad (4.20)$$

is defined by the relations:

$$\begin{matrix} R_{11} := -a + b + c & R_{12} := a - b + c & R_{13} := a + b - c \\ R_{21} := a + \alpha & R_{22} := b + \beta & R_{33} := c + \gamma \\ R_{31} := a - \alpha & R_{32} := b - \beta & R_{33} := c - \gamma \end{matrix} \quad (4.21)$$

Then the symmetries of the Clebsch-Gordan coefficients are encoded in the following:

Proposition IV.4

1. permutation of the rows:

$$\left\| \begin{matrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{matrix} \right\| = \epsilon_{ijk} \left\| \begin{matrix} R_{i1} & R_{i2} & R_{i3} \\ R_{j1} & R_{j2} & R_{j3} \\ R_{k1} & R_{k2} & R_{k3} \end{matrix} \right\| \quad \forall i, j, k \in \{1, 2, 3\} \quad (4.22)$$

2. permutation of the columns:

$$\left\| \begin{matrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{matrix} \right\| = \epsilon_{ijk} \left\| \begin{matrix} R_{1i} & R_{1j} & R_{1k} \\ R_{2i} & R_{2j} & R_{2k} \\ R_{3i} & R_{3j} & R_{3k} \end{matrix} \right\| \quad \forall i, j, k \in \{1, 2, 3\} \quad (4.23)$$

3. transposition:

$$\left\| \begin{matrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{matrix} \right\| = \left\| \begin{matrix} R_{11} & R_{21} & R_{31} \\ R_{12} & R_{22} & R_{32} \\ R_{13} & R_{23} & R_{33} \end{matrix} \right\| \quad (4.24)$$

Actually the proposition IV.1 may be derived simply by the proposition IV.4 applying to each of the Clebsch-Gordan coefficients appearing in it its transformation property under change of sign of all the angular momentum projection's (corresponding to the permutation of the second and the third rows of the R-symbol).

We strongly suspect that also the proposition IV.3 can be derived from the proposition IV.4 though at the present time we don't know how.

V. KRAMERS DEGENERATIONS, HIDDEN SYMMETRIES AND COUPLING THEORY

Proposition IV.2 was used by Wigner to show that the degeneracy of the energy levels of an odd number of electrons subjected to an arbitrary electric field could be derived from first principles as a particular case of the following:

Theorem V.1

Wigner's Theorem about Kramers degeneracy:
HP:

$$\text{H hamiltonian of a quantum system such that } [H,T]=0$$

$$H|\psi> = \ E|\psi> \tag{5.1}$$

$$\overleftarrow{\overleftarrow{|\psi>}} \ = \ -|\psi> \tag{5.2}$$

TH:

$$degeneration(E) \ \geq \ 2 \tag{5.3}$$

PROOF:

$$H\overleftarrow{|\psi>} \ = \ \overleftarrow{H|\psi>} \ = \ \overleftarrow{E|\psi>} \ = \ \bar{E}\overleftarrow{|\psi>} \tag{5.4}$$

Since H is self-adjoint it follows that $E \in \mathbb{R}$ and hence:

$$H\overleftarrow{|\psi>} \ = \ E\overleftarrow{|\psi>} \tag{5.5}$$

Since by the proposition IV.2:

$$<\psi\overleftarrow{|\psi>} \ = \ 0 \tag{5.6}$$

the thesis follows. ■

Let us now suppose to have a system of $n \in \mathbb{N} : n \geq 2$ quantum angular momenta $\vec{J}_1, \cdots, \vec{J}_n$ with an hamiltonian H having time-reversal and rotational symmetry, i.e. such that:

$$[H,T] \ = \ [H,\vec{J}^2] \ = \ [H,J_3] = \ 0 \tag{5.7}$$

Then:

$$H|\alpha, j_1, j_2, j_{12}, j_3, j_{123}, \cdots, j_{1\cdots n-1}, j_n, j, m> = \ E_{\alpha,j}|\alpha, j_1, j_2, j_{12}, j_3, j_{123}, \cdots, j_{1\cdots n-1}, j_n, j, m> \tag{5.8}$$

$$\vec{J}^2|\alpha, j_1, j_2, j_{12}, j_3, j_{123}, \cdots, j_{1\cdots n-1}, j_n, j, m> = \ j(j+1)|\alpha, j_1, j_2, j_{12}, j_3, j_{123}, \cdots, j_{1\cdots n-1}, j_n, j, m> \tag{5.9}$$

$$J_3|\alpha, j_1, j_2, j_{12}, j_3, j_{123}, \cdots, j_{1\cdots n-1}, j_n, j, m> = \ m|\alpha, j_1, j_2, j_{12}, j_3, j_{123}, \cdots, j_{1\cdots n-1}, j_n, j, m> \tag{5.10}$$

$$\overleftarrow{|\alpha, j_1, j_2, j_{12}, j_3, j_{123}, \cdots, j_{1\cdots n-1}, j_n, j, m>} \ = \ i^{2m}|\overleftarrow{\alpha}, j_1, j_2, j_{12}, j_3, j_{123}, \cdots, j_{1\cdots n-1}, j_n, j, -m> \tag{5.11}$$

$$\overleftarrow{\overleftarrow{|\alpha, j_1, j_2, j_{12}, j_3, j_{123}, \cdots, j_{1\cdots n-1}, j_n, j, m>}} \ = \ (-1)^{2j}|\overleftarrow{\overleftarrow{\alpha}}, j_1, j_2, j_{12}, j_3, j_{123}, \cdots, j_{1\cdots n-1}, j_n, j, m> \tag{5.12}$$

where we have denoted with the label α a suitable set of quantum numbers encoding every suppletive hidden symmetry.
Given $j \in \mathbb{H}$ let us observe that theorem V.1 can be applied if and only if $\overleftarrow{\overleftarrow{\alpha}} \ = \ \alpha$.

Remark V.1

Let us remark that the presence of hidden symmetries can lead to surprises both in implementing a *coupling scheme* and in Recoupling Theory where the transition between different *coupling schemes* is implemented, as we will show through some examples.

Example V.1

Let us consider $Z \in \mathbb{N} : n \geq 2$ bosons of spin zero and unary mass subjected to a force's field with Keplerian energy potential $V(r) := -\frac{1}{r}$ and hence having hamiltonian:

$$H := \sum_{I=1}^{Z} H_I \tag{5.13}$$

$$H_I := \frac{\vec{p}_I^2}{2} - \frac{1}{|\vec{x}_I|} \;\; I \in \{1, \cdots, Z\} \tag{5.14}$$

As it is well-known the Keplerian energy potential has a symmetry group bigger than the rotational symmetry group SO(3) of a generic central force's field; specifically such a symmetry group (easily obtained considering that, beside the angular momentum $\vec{L}$, there is a suppletive conserved quantity: the Laplace-Runge-Lenz vector) is:

- SO(4) for bound states
- SO(3,1) for free states

and is responsible of the closeness of the classical orbits (cfr. for instance the 14^{th} chapter "Dynamical Symmetries" of [74]).

Let us introduce the Laplace-Runge-Lenz operators:

$$\vec{M}_I := \frac{1}{2}(\vec{p}_I \wedge \vec{L}_I - \vec{L}_I \wedge \vec{p}_I) - \frac{\vec{x}_I}{|\vec{x}_I|} \;\; I \in \{1, \cdots, Z\} \tag{5.15}$$

where of course $\vec{L}_I := \vec{x}_I \wedge \vec{p}_I$ is the orbital angular momentum of the I^{th} electron.

Given an eigenvalue E_I corresponding to a bound state of H_I let us introduce the rescaled Laplace-Runge-Lenz operators:

$$\vec{M}'_I := \sqrt{-\frac{1}{2E_I}} \vec{M}_I \;\; I \in \{1, \cdots, Z\} \tag{5.16}$$

The Lie algebra generated by $\{\vec{L}_I, \vec{M}'_I\}_{I=1}^{Z}$ is completely determined by the following relations:

$$[L_{Ii}, L_{Jj}] = i\delta_{IJ}\epsilon_{ijk}L_{Ik} \;\; \forall I, J \in \{1, \cdots, Z\}, \forall i, j \in \{1, 2, 3\} \tag{5.17}$$

$$[M'_{Ii}, M'_{Jj}] = i\delta_{IJ}\epsilon_{ijk}L_{Ik} \;\; \forall I, J \in \{1, \cdots, Z\}, \forall i, j \in \{1, 2, 3\} \tag{5.18}$$

$$[M'_{Ii}, L_{Jj}] = i\delta_{IJ}\epsilon_{ijk}M'_{Ik} \;\; \forall I, J \in \{1, \cdots, Z\}, \forall i, j \in \{1, 2, 3\} \tag{5.19}$$

that together with:

$$[H, L_{Ii}] = [H, M'_{Ii}] = 0 \;\; \forall I, J \in \{1, \cdots, Z\}, \forall i \in \{1, 2, 3\} \tag{5.20}$$

describe the $\otimes_{I=1}^{Z} SO(4)$ symmetry of H.

Introduced the operators:

$$\vec{J}_{(1)I} := \frac{1}{2}(\vec{L}_I + \vec{M}'_I) \;\; I \in \{1, \cdots, Z\} \tag{5.21}$$

$$\vec{J}_{(2)I} := \frac{1}{2}(\vec{L}_I - \vec{M}'_I) \;\; I \in \{1, \cdots, Z\} \tag{5.22}$$

the equation 5.17, the equation 5.18 and the equation 5.19 imply that:

$$[J_{(\alpha)Ii}, J_{(\beta)Jj}] \; = \; i\delta_{\alpha\beta}\delta_{IJ}\epsilon_{ijk}J_{(\alpha)Ik} \;\; \forall\alpha,\beta \in \{1,2\}, \forall I, J \in \{1,\cdots,Z\}, \forall i,j \in \{1,2,3\} \tag{5.23}$$

$$[H, J_{(\alpha)Ii}] \; = \; 0 \;\; \forall\alpha \in \{1,2\}, \forall I \in \{1,\cdots,Z\}, \forall i \in \{1,2,3\} \tag{5.24}$$

The rank [17] of $\otimes_{I=1}^{Z} SO(4)$ is clearly 2 Z.

Let us now recall the following basic theorem (see for instance the section 3.6 "Theorem of Racah" and the chapter 16 "Proof of Racah's Theorem" of [74]) proved by Giulio Racah in 1950:

Theorem V.2

Racah's Theorem:

The number of linearly independent Casimir operators of a semisimple Lie group is equal to its rank

According to the theorem V.2 it follows that $\otimes_{I=1}^{Z} SO(4)$ has 2 Z linearly independent Casimir operators that may be chosen to be $\{\vec{J}^2_{(1)I}, \vec{J}^2_{(2)I} \; I \in \{1,\cdots,Z\}\}$ whose eigenvalues may be used to characterize the multiplets [18].

Since the equation 5.23 shows that such operators are 2 Z uncoupled quantum angular momenta it follows that:

$$\begin{aligned} &\vec{J}^2_{(1)I}|j_{(1),1}, m_{(1),1}, \cdots, j_{(1),Z}, m_{(1),Z}, j_{(2),1}, m_{(2),1}, \cdots, j_{(2),Z}, m_{(2),Z} > = \\ &\qquad j_{(1)I}(j_{(1)I}+1)|j_{(1),1}, m_{(1),1}, \cdots, j_{(1),Z}, m_{(1),Z}, j_{(2),1}, m_{(2),1}, \cdots, j_{(2),Z}, m_{(2),Z} > \\ &\qquad\qquad \forall m_{(i),I} \in \{-j_{(i)I}, \cdots, j_{(i)I}\}, \; \forall j_{(i)I} \in \frac{\mathbb{N}}{2}, \forall i \in \{1,2\}, \forall I \in \{1,\cdots,Z\} \end{aligned} \tag{5.25}$$

$$\begin{aligned} &J_{(1)I3}|j_{(1),1}, m_{(1),1}, \cdots, j_{(1),Z}, m_{(1),Z}, j_{(2),1}, m_{(2),1}, \cdots, j_{(2),Z}, m_{(2),Z} > = \\ &\qquad m_{(1),I}|j_{(1),1}, m_{(1),1}, \cdots, j_{(1),Z}, m_{(1),Z}, j_{(2),1}, m_{(2),1}, \cdots, j_{(2),Z}, m_{(2),Z} > \\ &\qquad\qquad \forall m_{(i),I} \in \{-j_{(i)I}, \cdots, j_{(i)I}\}, \; \forall j_{(i)I} \in \frac{\mathbb{N}}{2}, \forall i \in \{1,2\}, \forall I \in \{1,\cdots,Z\} \end{aligned} \tag{5.26}$$

$$\begin{aligned} &\vec{J}^2_{(2)I}|j_{(1),1}, m_{(1),1}, \cdots, j_{(1),Z}, m_{(1),Z}, j_{(2),1}, m_{(2),1}, \cdots, j_{(2),Z}, m_{(2),Z} > = \\ &\qquad j_{(2)I}(j_{(2)I}+1)|j_{(1),1}, m_{(1),1}, \cdots, j_{(1),Z}, m_{(1),Z}, j_{(2),1}, m_{(2),1}, \cdots, j_{(2),Z}, m_{(2),Z} > \\ &\qquad\qquad \forall m_{(i),I} \in \{-j_{(i)I}, \cdots, j_{(i)I}\}, \; \forall j_{(i)I} \in \frac{\mathbb{N}}{2}, \forall i \in \{1,2\}, \forall I \in \{1,\cdots,Z\} \end{aligned} \tag{5.27}$$

$$\begin{aligned} &J_{(2)I3}|j_{(1),1}, m_{(1),1}, \cdots, j_{(1),Z}, m_{(1),Z}, j_{(2),1}, m_{(2),1}, \cdots, j_{(2),Z}, m_{(2),Z} > = \\ &\qquad m_{(2),I}|j_{(1),1}, m_{(1),1}, \cdots, j_{(1),Z}, m_{(1),Z}, j_{(2),1}, m_{(2),1}, \cdots, j_{(2),Z}, m_{(2),Z} > \\ &\qquad\qquad \forall m_{(i),I} \in \{-j_{(i)I}, \cdots, j_{(i)I}\}, \; \forall j_{(i)I} \in \frac{\mathbb{N}}{2}, \forall i \in \{1,2\}, \forall I \in \{1,\cdots,Z\} \end{aligned} \tag{5.28}$$

We can, in an equivalent way, to take into account the set of Casimir operators $\{C_{(1)I}, C_{(2)I} \; I \in \{1,\cdots,Z\}\}$ where:

$$C_{(1)I} \; := \; \vec{J}^2_{(1)I} + \vec{J}^2_{(2)I} \;\; I \in \{1,\cdots,Z\} \tag{5.29}$$

$$C_{(2)I} \; := \; \vec{J}^2_{(1)I} - \vec{J}^2_{(2)I} \;\; I \in \{1,\cdots,Z\} \tag{5.30}$$

[17] defined as the maximal number of commuting generators.
[18] defined as the irreducible invariant subspaces.

Since:

$$\vec{M}_I \cdot \vec{L}_I \ = \ 0 \ \ \forall I \in \{1, \cdots , Z\} \tag{5.31}$$

it follows that:

$$C_{(2)I} \ = \ 0 \ \ \forall I \in \{1, \cdots , Z\} \tag{5.32}$$

so that we can consider only the subspace such that $j_{(1)I} = j_{(2)I} \ \forall I \in \{1, \cdots , Z\}$.

Taking into account the equation 5.16, the equation 5.21 and the equation 5.22 it follows that:

$$H|j_{(1),1}, m_{(1),1}, \cdots , j_{(1),Z}, m_{(1),Z}, j_{(1),1}, m_{(2),1}, \cdots , j_{(1),Z}, m_{(2),Z} > = \sum_{I=1}^{Z} E_{j_{(1),I}} |j_{(1),1}, m_{(1),1}, \cdots , j_{(1),Z}, m_{(1),Z}, j_{(1),1}, m_{(2),1}, \cdots , j_{(1),Z}, m_{(2),Z} > \tag{5.33}$$

where:

$$E_{j_{(1),I}} \ := \ - \frac{j_{(1),I}^2}{(2 j_{(1),I} + 1)^2} \quad j_{(1),I} \in \frac{\mathbb{N}}{2} , \, I \in \{1, \cdots , Z\} \tag{5.34}$$

Let us now observe that since $\{\vec{J}_{(1),I}, \vec{J}_{(2),I} \ \ I \in \{1, \cdots , Z\}\}$ are $2Z$ uncoupled angular momentum operators, the equation 3.10 implies that:

$$T \vec{J}_{(i),I} T^{-1} \ = \ - \vec{J}_{(i),I} \ \ \forall I \in \{1, \cdots , Z\}, \forall i \in \{1, 2\} \tag{5.35}$$

while the equation 3.11 and the equation 3.12 imply, respectively, that:

$$\overleftarrow{|j_{(1),1}, m_{(1),1}, \cdots , j_{(1),Z}, m_{(1),Z}, j_{(2),1}, m_{(2),1}, \cdots , j_{(2),Z}, m_{(2),Z} >} \ = \ i^{2 \sum_{I=1}^{Z} (m_{(1),I} + m_{(2),I})} |j_{(1),1}, -m_{(1),1}, \cdots , j_{(1),Z}, -m_{(1),Z}, j_{(2),1}, -m_{(2),1}, \cdots , j_{(2),Z}, -m_{(2),Z} > \tag{5.36}$$

and:

$$\overleftarrow{\overleftarrow{|j_{(1),1}, m_{(1),1}, \cdots , j_{(1),Z}, m_{(1),Z}, j_{(2),1}, m_{(2),1}, \cdots , j_{(2),Z}, m_{(2),Z} >}} \ = \ (-1)^{2 \sum_{I=1}^{Z} (j_{(1),I} + j_{(2),I})} |j_{(1),1}, m_{(1),1}, \cdots , j_{(1),Z}, m_{(1),Z}, j_{(2),1}, m_{(2),1}, \cdots , j_{(2),Z}, m_{(2),Z} > \tag{5.37}$$

Let us now consider the coupling of the quantum angular momenta operators $\{\vec{J}_{(i)I} \ I \in \{1, \cdots , Z\}, i \in \{1, 2\}\}$.

Owing to the proposition II.3 it follows that:

$$\begin{aligned} & |j_{(1)1}, j_{(1)2}, j_{(1)12}, j_{(1)3}, j_{(1)123}, \cdots , j_{(1)1 \cdots Z-1}, j_{(1)Z}, j_{(1)}, m_{(1)}, j_{(2)1}, j_{(2)2}, j_{(2)12}, j_{(2)3}, j_{(2)123}, \cdots , j_{(2)1 \cdots Z-1}, j_{(2)Z}, j_{(2)}, m_{(2)} > \\ & = \prod_{i \in \{1,2\}} \sum_{m_{(i)1} = -j_{(i)1}}^{j_{(i)1}} \sum_{m_{(i)2} = -j_{(i)2}}^{j_{(i)2}} \cdots \sum_{m_{(i)Z} = -j_{(i)Z}}^{j_{(i)Z}} C^{j_{(i)} m_{(i)}}_{j_{(i)1 \cdots Z-1} m_{(i)1 \cdots Z-1} j_{(i)Z} m_{(i)Z}} \cdots C^{j_{(i)123} m_{(i)123}}_{j_{(i)12} m_{(i)12} j_{(i)3} m_{(i)3}} C^{j_{(i)12} m_{(i)12}}_{j_{(i)1} m_{(i)1} j_{(i)2} m_{(i)2}} \\ & |j_{(1)1}, m_{(1)1}, \cdots , j_{(1)Z}, m_{(1)Z}, j_{(2)1}, m_{(2)1}, \cdots , j_{(2)Z}, m_{(2)Z} > \\ & \forall m_{(i)} \in \{-j_{(i)}, \cdots , j_{(i)}\}, \forall j_{(i)} \in \{|j_{(i)1 \cdots Z-1} - j_{(i)Z}|, \cdots , j_{(i)1 \cdots Z-1} + j_{(i)Z}\} \\ & \forall j_{(i)1 \cdots Z-1} \in \{|j_{(i)1 \cdots Z-2} - j_{(i)Z-1}|, \cdots , j_{(i)1 \cdots Z-2} + j_{(i)Z-1}\}, \cdots , \forall j_{(i)123} \in \{|j_{(i)12} - j_{(i)3}|, \cdots , j_{(i)12} + j_{(i)3}\}, \\ & \forall j_{(i)12} \in \{|j_{(i)1} - j_{(i)2}|, \cdots , j_{(i)1} + j_{(i)2}\}, \forall i \in \{1, 2\} \end{aligned} \tag{5.38}$$

The equation 3.11 and the equation 3.12 imply, respectively, that:

$$\begin{aligned} & \overleftarrow{|j_{(1)1}, j_{(1)2}, j_{(1)12}, j_{(1)3}, j_{(1)123}, \cdots , j_{(1)1 \cdots Z-1}, j_{(1)Z}, j_{(1)}, m_{(1)}, j_{(2)1}, j_{(2)2}, j_{(2)12}, j_{(2)3}, j_{(2)123}, \cdots , j_{(2)1 \cdots Z-1}, j_{(2)Z}, j_{(2)}, m_{(2)} >} \\ & = \ i^{2(m_{(1)} + m_{(2)})} \\ & |j_{(1)1}, j_{(1)2}, j_{(1)12}, j_{(1)3}, j_{(1)123}, \cdots , j_{(1)1 \cdots Z-1}, j_{(1)Z}, j_{(1)}, -m_{(1)}, j_{(2)1}, j_{(2)2}, j_{(2)12}, j_{(2)3}, j_{(2)123}, \cdots , j_{(2)1 \cdots Z-1}, j_{(2)Z}, j_{(2)}, -m_{(2)} > \end{aligned} \tag{5.39}$$

$$\overleftarrow{\overleftarrow{|j_{(1)1}, j_{(1)2}, j_{(1)12}, j_{(1)3}, j_{(1)123}, \cdots, j_{(1)1\cdots Z-1}, j_{(1)Z}, j_{(1)}, m_{(1)}, j_{(2)1}, j_{(2)2}, j_{(2)12}, j_{(2)3}, j_{(2)123}, \cdots, j_{(2)1\cdots Z-1}, j_{(2)Z}, j_{(2)}, m_{(2)} >}}$$
$$= (-1)^{2(j_{(1)}+j_{(2)})}$$
$$|j_{(1)1}, j_{(1)2}, j_{(1)12}, j_{(1)3}, j_{(1)123}, \cdots, j_{(1)1\cdots Z-1}, j_{(1)Z}, j_{(1)}, m_{(1)}, j_{(2)1}, j_{(2)2}, j_{(2)12}, j_{(2)3}, j_{(2)123}, \cdots, j_{(2)1\cdots Z-1}, j_{(2)Z}, j_{(2)}, m_{(2)} > \quad (5.40)$$

Let us now apply the time reversal operator to the equation 5.38; we obtain that:

$$\overleftarrow{|j_{(1)1}, j_{(1)2}, j_{(1)12}, j_{(1)3}, j_{(1)123}, \cdots, j_{(1)1\cdots Z-1}, j_{(1)Z}, j_{(1)}, m_{(1)}, j_{(2)1}, j_{(2)2}, j_{(2)12}, j_{(2)3}, j_{(2)123}, \cdots, j_{(2)1\cdots Z-1}, j_{(2)Z}, j_{(2)}, m_{(2)} >}$$
$$= \prod_{i \in \{1,2\}} \sum_{m_{(i)1}=-j_{(i)1}}^{j_{(i)1}} \sum_{m_{(i)2}=-j_{(i)2}}^{j_{(i)2}} \cdots \sum_{m_{(i)Z}=-j_{(i)Z}}^{j_{(i)Z}} \overline{C^{j_{(i)} m_{(i)}}_{j_{(i)1\cdots Z-1} m_{(i)1\cdots Z-1} j_{(i)Z} m_{(i)Z}} \cdots C^{j_{(i)123} m_{(i)123}}_{j_{(i)12} m_{(i)12} j_{(i)3} m_{(i)3}} C^{j_{(i)12} m_{(i)12}}_{j_{(i)1} m_{(i)1} j_{(i)2} m_{(i)2}}}$$
$$\overleftarrow{|j_{(1)1}, m_{(1)1}, \cdots, j_{(1)Z}, m_{(1)Z}, j_{(2)1}, m_{(2)1}, \cdots, j_{(2)Z}, m_{(2)Z} >}$$
$$\forall m_{(i)} \in \{-j_{(i)}, \cdots, j_{(i)}\}, \forall j_{(i)} \in \{|j_{(i)1\cdots Z-1} - j_{(i)Z}|, \cdots, j_{(i)1\cdots Z-1} + j_{(i)Z}\}$$
$$\forall j_{(i)1\cdots Z-1} \in \{|j_{(i)1\cdots Z-2} - j_{(i)Z-1}|, \cdots, j_{(i)1\cdots Z-2} + j_{(i)Z-1}\}, \cdots, \forall j_{(i)123} \in \{|j_{(i)12} - j_{(i)3}|, \cdots, j_{(i)12} + j_{(i)3}\},$$
$$\forall j_{(i)12} \in \{|j_{(i)1} - j_{(i)2}|, \cdots, j_{(i)1} + j_{(i)2}\}, \forall i \in \{1,2\} \quad (5.41)$$

$$\overleftarrow{|j_{(1)1}, j_{(1)2}, j_{(1)12}, j_{(1)3}, j_{(1)123}, \cdots, j_{(1)1\cdots Z-1}, j_{(1)Z}, j_{(1)}, m_{(1)}, j_{(2)1}, j_{(2)2}, j_{(2)12}, j_{(2)3}, j_{(2)123}, \cdots, j_{(2)1\cdots Z-1}, j_{(2)Z}, j_{(2)}, m_{(2)} >}$$
$$= \prod_{i \in \{1,2\}} \sum_{m_{(i)1}=-j_{(i)1}}^{j_{(i)1}} \sum_{m_{(i)2}=-j_{(i)2}}^{j_{(i)2}} \cdots \sum_{m_{(i)Z}=-j_{(i)Z}}^{j_{(i)Z}} C^{j_{(i)} m_{(i)}}_{j_{(i)1\cdots Z-1} m_{(i)1\cdots Z-1} j_{(i)Z} m_{(i)Z}} \cdots C^{j_{(i)123} m_{(i)123}}_{j_{(i)12} m_{(i)12} j_{(i)3} m_{(i)3}} C^{j_{(i)12} m_{(i)12}}_{j_{(i)1} m_{(i)1} j_{(i)2} m_{(i)2}}$$
$$\overleftarrow{|j_{(1)1}, m_{(1)1}, \cdots, j_{(1)Z}, m_{(1)Z}, j_{(2)1}, m_{(2)1}, \cdots, j_{(2)Z}, m_{(2)Z} >}$$
$$\forall m_{(i)} \in \{-j_{(i)}, \cdots, j_{(i)}\}, \forall j_{(i)} \in \{|j_{(i)1\cdots Z-1} - j_{(i)Z}|, \cdots, j_{(i)1\cdots Z-1} + j_{(i)Z}\}$$
$$\forall j_{(i)1\cdots Z-1} \in \{|j_{(i)1\cdots Z-2} - j_{(i)Z-1}|, \cdots, j_{(i)1\cdots Z-2} + j_{(i)Z-1}\}, \cdots, \forall j_{(i)123} \in \{|j_{(i)12} - j_{(i)3}|, \cdots, j_{(i)12} + j_{(i)3}\},$$
$$\forall j_{(i)12} \in \{|j_{(i)1} - j_{(i)2}|, \cdots, j_{(i)1} + j_{(i)2}\}, \forall i \in \{1,2\} \quad (5.42)$$

where we have used the anti-linearity of T and the reality of the Clebsch-Gordan coefficients. Using the equation 5.36 it follows that:

$$\overleftarrow{|j_{(1)1}, j_{(1)2}, j_{(1)12}, j_{(1)3}, j_{(1)123}, \cdots, j_{(1)1\cdots Z-1}, j_{(1)Z}, j_{(1)}, m_{(1)}, j_{(2)1}, j_{(2)2}, j_{(2)12}, j_{(2)3}, j_{(2)123}, \cdots, j_{(2)1\cdots Z-1}, j_{(2)Z}, j_{(2)}, m_{(2)} >}$$
$$= \prod_{i \in \{1,2\}} \sum_{m_{(i)1}=-j_{(i)1}}^{j_{(i)1}} \sum_{m_{(i)2}=-j_{(i)2}}^{j_{(i)2}} \cdots \sum_{m_{(i)Z}=-j_{(i)Z}}^{j_{(i)Z}} C^{j_{(i)} m_{(i)}}_{j_{(i)1\cdots Z-1} m_{(i)1\cdots Z-1} j_{(i)Z} m_{(i)Z}} \cdots C^{j_{(i)123} m_{(i)123}}_{j_{(i)12} m_{(i)12} j_{(i)3} m_{(i)3}} C^{j_{(i)12} m_{(i)12}}_{j_{(i)1} m_{(i)1} j_{(i)2} m_{(i)2}}$$
$$i^{2\sum_{I=1}^{Z}(m_{(1),I}+m_{(2),I})} |j_{(1)1}, -m_{(1)1}, \cdots, j_{(1)Z}, -m_{(1)Z}, j_{(2)1}, -m_{(2)1}, \cdots, j_{(2)Z}, -m_{(2)Z} >$$
$$\forall m_{(i)} \in \{-j_{(i)}, \cdots, j_{(i)}\}, \forall j_{(i)} \in \{|j_{(i)1\cdots Z-1} - j_{(i)Z}|, \cdots, j_{(i)1\cdots Z-1} + j_{(i)Z}\}$$
$$\forall j_{(i)1\cdots Z-1} \in \{|j_{(i)1\cdots Z-2} - j_{(i)Z-1}|, \cdots, j_{(i)1\cdots Z-2} + j_{(i)Z-1}\}, \cdots, \forall j_{(i)123} \in \{|j_{(i)12} - j_{(i)3}|, \cdots, j_{(i)12} + j_{(i)3}\},$$
$$\forall j_{(i)12} \in \{|j_{(i)1} - j_{(i)2}|, \cdots, j_{(i)1} + j_{(i)2}\}, \forall i \in \{1,2\} \quad (5.43)$$

Comparing the equation 5.36 and the equation 5.43 it follows that:

$$|j_{(1)1}, j_{(1)2}, j_{(1)12}, j_{(1)3}, j_{(1)123}, \cdots, j_{(1)1\cdots Z-1}, j_{(1)Z}, j_{(1)}, -m_{(1)}, j_{(2)1}, j_{(2)2}, j_{(2)12}, j_{(2)3}, j_{(2)123}, \cdots, j_{(2)1\cdots Z-1}, j_{(2)Z}, j_{(2)}, -m_{(2)} >$$
$$= \prod_{i \in \{1,2\}} \sum_{m_{(i)1}=-j_{(i)1}}^{j_{(i)1}} \sum_{m_{(i)2}=-j_{(i)2}}^{j_{(i)2}} \cdots \sum_{m_{(i)Z}=-j_{(i)Z}}^{j_{(i)Z}} C^{j_{(i)} m_{(i)}}_{j_{(i)1\cdots Z-1} m_{(i)1\cdots Z-1} j_{(i)Z} m_{(i)Z}} \cdots C^{j_{(i)123} m_{(i)123}}_{j_{(i)12} m_{(i)12} j_{(i)3} m_{(i)3}} C^{j_{(i)12} m_{(i)12}}_{j_{(i)1} m_{(i)1} j_{(i)2} m_{(i)2}}$$
$$i^{2[m_{(1)}+m_{(2)}-\sum_{I=1}^{Z}(m_{(1),I}+m_{(2),I})]} |j_{(1)1}, -m_{(1)1}, \cdots, j_{(1)Z}, -m_{(1)Z}, j_{(2)1}, -m_{(2)1}, \cdots, j_{(2)Z}, -m_{(2)Z} >$$
$$\forall m_{(i)} \in \{-j_{(i)}, \cdots, j_{(i)}\}, \forall j_{(i)} \in \{|j_{(i)1\cdots Z-1} - j_{(i)Z}|, \cdots, j_{(i)1\cdots Z-1} + j_{(i)Z}\}$$
$$\forall j_{(i)1\cdots Z-1} \in \{|j_{(i)1\cdots Z-2} - j_{(i)Z-1}|, \cdots, j_{(i)1\cdots Z-2} + j_{(i)Z-1}\}, \cdots, \forall j_{(i)123} \in \{|j_{(i)12} - j_{(i)3}|, \cdots, j_{(i)12} + j_{(i)3}\},$$
$$\forall j_{(i)12} \in \{|j_{(i)1} - j_{(i)2}|, \cdots, j_{(i)1} + j_{(i)2}\}, \forall i \in \{1,2\} \quad (5.44)$$

that, using the proposition II.1, implies that:

Proposition V.1

$$|j_{(1)1}, j_{(1)2}, j_{(1)12}, j_{(1)3}, j_{(1)123}, \cdots, j_{(1)1\cdots Z-1}, j_{(1)Z}, j_{(1)}, -m_{(1)}, j_{(2)1}, j_{(2)2}, j_{(2)12}, j_{(2)3}, j_{(2)123}, \cdots, j_{(2)1\cdots Z-1}, j_{(2)Z}, j_{(2)}, -m_{(2)} >$$
$$= \prod_{i\in\{1,2\}} \sum_{m_{(i)1}=-j_{(i)1}}^{j_{(i)1}} \sum_{m_{(i)2}=-j_{(i)2}}^{j_{(i)2}} \cdots \sum_{m_{(i)Z}=-j_{(i)Z}}^{j_{(i)Z}} C^{j_{(i)}m_{(i)}}_{j_{(i)1\cdots Z-1}m_{(i)1\cdots Z-1}j_{(i)Z}m_{(i)Z}} \cdots C^{j_{(i)123}m_{(i)123}}_{j_{(i)12}m_{(i)12}j_{(i)3}m_{(i)3}} C^{j_{(i)12}m_{(i)12}}_{j_{(i)1}m_{(i)1}j_{(i)2}m_{(i)2}}$$
$$|j_{(1)1}, -m_{(1)1}, \cdots, j_{(1)Z}, -m_{(1)Z}, j_{(2)1}, -m_{(2)1}, \cdots, j_{(2)Z}, -m_{(2)Z} >$$
$$\forall m_{(i)} \in \{-j_{(i)}, \cdots, j_{(i)}\}, \forall j_{(i)} \in \{|j_{(i)1\cdots Z-1} - j_{(i)Z}|, \cdots, j_{(i)1\cdots Z-1} + j_{(i)Z}\}$$
$$\forall j_{(i)1\cdots Z-1} \in \{|j_{(i)1\cdots Z-2} - j_{(i)Z-1}|, \cdots, j_{(i)1\cdots Z-2} + j_{(i)Z-1}\}, \cdots, \forall j_{(i)123} \in \{|j_{(i)12} - j_{(i)3}|, \cdots, j_{(i)12} + j_{(i)3}\},$$
$$\forall j_{(i)12} \in \{|j_{(i)1} - j_{(i)2}|, \cdots, j_{(i)1} + j_{(i)2}\}, \forall i \in \{1,2\} \quad (5.45)$$

Let us now apply the squared time reversal operator to the equation 5.38; we obtain that:

$$\overleftarrow{\overleftarrow{|j_{(1)1}, j_{(1)2}, j_{(1)12}, j_{(1)3}, j_{(1)123}, \cdots, j_{(1)1\cdots Z-1}, j_{(1)Z}, j_{(1)}, m_{(1)}, j_{(2)1}, j_{(2)2}, j_{(2)12}, j_{(2)3}, j_{(2)123}, \cdots, j_{(2)1\cdots Z-1}, j_{(2)Z}, j_{(2)}, m_{(2)} >}}$$
$$= \prod_{i\in\{1,2\}} \sum_{m_{(i)1}=-j_{(i)1}}^{j_{(i)1}} \sum_{m_{(i)2}=-j_{(i)2}}^{j_{(i)2}} \cdots \sum_{m_{(i)Z}=-j_{(i)Z}}^{j_{(i)Z}} C^{j_{(i)}m_{(i)}}_{j_{(i)1\cdots Z-1}m_{(i)1\cdots Z-1}j_{(i)Z}m_{(i)Z}} \cdots C^{j_{(i)123}m_{(i)123}}_{j_{(i)12}m_{(i)12}j_{(i)3}m_{(i)3}} C^{j_{(i)12}m_{(i)12}}_{j_{(i)1}m_{(i)1}j_{(i)2}m_{(i)2}}$$
$$\overleftarrow{\overleftarrow{|j_{(1)1}, m_{(1)1}, \cdots, j_{(1)Z}, m_{(1)Z}, j_{(2)1}, m_{(2)1}, \cdots, j_{(2)Z}, m_{(2)Z} >}}$$
$$\forall m_{(i)} \in \{-j_{(i)}, \cdots, j_{(i)}\}, \forall j_{(i)} \in \{|j_{(i)1\cdots Z-1} - j_{(i)Z}|, \cdots, j_{(i)1\cdots Z-1} + j_{(i)Z}\}$$
$$\forall j_{(i)1\cdots Z-1} \in \{|j_{(i)1\cdots Z-2} - j_{(i)Z-1}|, \cdots, j_{(i)1\cdots Z-2} + j_{(i)Z-1}\}, \cdots, \forall j_{(i)123} \in \{|j_{(i)12} - j_{(i)3}|, \cdots, j_{(i)12} + j_{(i)3}\},$$
$$\forall j_{(i)12} \in \{|j_{(i)1} - j_{(i)2}|, \cdots, j_{(i)1} + j_{(i)2}\}, \forall i \in \{1,2\} \quad (5.46)$$

where we have used the linearity of T^2.
Using the equation 5.37 it follows that:

$$\overleftarrow{\overleftarrow{|j_{(1)1}, j_{(1)2}, j_{(1)12}, j_{(1)3}, j_{(1)123}, \cdots, j_{(1)1\cdots Z-1}, j_{(1)Z}, j_{(1)}, m_{(1)}, j_{(2)1}, j_{(2)2}, j_{(2)12}, j_{(2)3}, j_{(2)123}, \cdots, j_{(2)1\cdots Z-1}, j_{(2)Z}, j_{(2)}, m_{(2)} >}}$$
$$= \prod_{i\in\{1,2\}} \sum_{m_{(i)1}=-j_{(i)1}}^{j_{(i)1}} \sum_{m_{(i)2}=-j_{(i)2}}^{j_{(i)2}} \cdots \sum_{m_{(i)Z}=-j_{(i)Z}}^{j_{(i)Z}} C^{j_{(i)}m_{(i)}}_{j_{(i)1\cdots Z-1}m_{(i)1\cdots Z-1}j_{(i)Z}m_{(i)Z}} \cdots C^{j_{(i)123}m_{(i)123}}_{j_{(i)12}m_{(i)12}j_{(i)3}m_{(i)3}} C^{j_{(i)12}m_{(i)12}}_{j_{(i)1}m_{(i)1}j_{(i)2}m_{(i)2}}$$
$$(-1)^{2\sum_{I=1}^{Z}(j_{(1)I}+j_{(2)I})}|j_{(1)1}, m_{(1)1}, \cdots, j_{(1)Z}, m_{(1)Z}, j_{(2)1}, m_{(2)1}, \cdots, j_{(2)Z}, m_{(2)Z} >$$
$$\forall m_{(i)} \in \{-j_{(i)}, \cdots, j_{(i)}\}, \forall j_{(i)} \in \{|j_{(i)1\cdots Z-1} - j_{(i)Z}|, \cdots, j_{(i)1\cdots Z-1} + j_{(i)Z}\}$$
$$\forall j_{(i)1\cdots Z-1} \in \{|j_{(i)1\cdots Z-2} - j_{(i)Z-1}|, \cdots, j_{(i)1\cdots Z-2} + j_{(i)Z-1}\}, \cdots, \forall j_{(i)123} \in \{|j_{(i)12} - j_{(i)3}|, \cdots, j_{(i)12} + j_{(i)3}\},$$
$$\forall j_{(i)12} \in \{|j_{(i)1} - j_{(i)2}|, \cdots, j_{(i)1} + j_{(i)2}\}, \forall i \in \{1,2\} \quad (5.47)$$

Comparing the equation 5.37 and the equation 5.47 it follows that:

Proposition V.2

$$|j_{(1)1}, j_{(1)2}, j_{(1)12}, j_{(1)3}, j_{(1)123}, \cdots, j_{(1)1\cdots Z-1}, j_{(1)Z}, j_{(1)}, m_{(1)}, j_{(2)1}, j_{(2)2}, j_{(2)12}, j_{(2)3}, j_{(2)123}, \cdots, j_{(2)1\cdots Z-1}, j_{(2)Z}, j_{(2)}, m_{(2)} >$$
$$= \prod_{i\in\{1,2\}} \sum_{m_{(i)1}=-j_{(i)1}}^{j_{(i)1}} \sum_{m_{(i)2}=-j_{(i)2}}^{j_{(i)2}} \cdots \sum_{m_{(i)Z}=-j_{(i)Z}}^{j_{(i)Z}} C^{j_{(i)}m_{(i)}}_{j_{(i)1\cdots Z-1}m_{(i)1\cdots Z-1}j_{(i)Z}m_{(i)Z}} \cdots C^{j_{(i)123}m_{(i)123}}_{j_{(i)12}m_{(i)12}j_{(i)3}m_{(i)3}} C^{j_{(i)12}m_{(i)12}}_{j_{(i)1}m_{(i)1}j_{(i)2}m_{(i)2}}$$
$$(-1)^{2(\sum_{I=1}^{Z}(j_{(1)I}+j_{(2)I})-j_{(1)}-j_{2})}|j_{(1)1}, m_{(1)1}, \cdots, j_{(1)Z}, m_{(1)Z}, j_{(2)1}, m_{(2)1}, \cdots, j_{(2)Z}, m_{(2)Z} >$$
$$\forall m_{(i)} \in \{-j_{(i)}, \cdots, j_{(i)}\}, \forall j_{(i)} \in \{|j_{(i)1\cdots Z-1} - j_{(i)Z}|, \cdots, j_{(i)1\cdots Z-1} + j_{(i)Z}\}$$
$$\forall j_{(i)1\cdots Z-1} \in \{|j_{(i)1\cdots Z-2} - j_{(i)Z-1}|, \cdots, j_{(i)1\cdots Z-2} + j_{(i)Z-1}\}, \cdots, \forall j_{(i)123} \in \{|j_{(i)12} - j_{(i)3}|, \cdots, j_{(i)12} + j_{(i)3}\},$$
$$\forall j_{(i)12} \in \{|j_{(i)1} - j_{(i)2}|, \cdots, j_{(i)1} + j_{(i)2}\}, \forall i \in \{1,2\} \quad (5.48)$$

Let us now remark that, exactly as in the analogous computation performed in the section III, the equation 5.38 and the proposition V.2 are consistent if and only if:

$$(-1)^{2(\sum_{I=1}^{Z}(j_{(1)I}+j_{(2)I})-j_{(1)}-j_{2})} = 1$$
$$\forall j_{(1)} \in \{\mathcal{J}_{min}(j_{(1)1}, \cdots, j_{(1)Z}), \cdots, \mathcal{J}_{max}(j_{(1)1}, \cdots, j_{(1)Z})\}, \forall j_{(2)} \in \{\mathcal{J}_{min}(j_{(2)1}, \cdots, j_{(2)Z}), \cdots, \mathcal{J}_{max}(j_{(2)1}, \cdots, j_{(2)Z})\},$$
$$\forall j_{iI} \in \frac{\mathbb{N}}{2}, \forall i \in \{1,2\}, \forall I \in \{1, \cdots, Z\} \quad (5.49)$$

Again the equation 5.49 is, anyway, a trivial consequence of the proposition III.1.

Let us at last analyze the degeneracy of the energy levels.
We have obviously that:

$$H|j_{(1)1}, m_{(1)1}, \cdots, j_{(1)Z}, m_{(1)Z}, j_{(2)1} := j_{(1)1}, m_{(2)1}, \cdots, j_{(2)Z} := j_{(1)Z}, m_{(2)Z} > = E_{j_{(1),1}, \cdots, j_{(1),Z}} |j_{(1)1}, m_{(1)1}, \cdots, j_{(1)Z}, m_{(1)Z}, j_{(2)1} := j_{(1)1}, m_{(2)1}, \cdots, j_{(2)Z} := j_{(1)Z}, m_{(2)Z} > \quad (5.50)$$

where:

$$E_{j_{(1),1}, \cdots, j_{(1),Z}} \; := \; \sum_{I=1}^{Z} E_{j_{(1),I}} \quad (5.51)$$

so that:

$$degeneration(E_{j_{(1),1}, \cdots, j_{(1),Z}}) \; = \; 2 \prod_{I=1}^{Z} (2 j_{(1),I} + 1) \quad (5.52)$$

where we recall that $E_{j_{(1),I}}$ is defined in the equation 5.34 and when the factor 2 in the right hand side of the equation 5.52 is owed to the fact that $j_{(2),I}$ is equal to $j_{(1)I}$ but, in general $m_{(2),I}$ is not equal to $m_{(1),I}$.

Such a degeneration cannot, anyway, be inferred from the theorem V.1 since the angular momentum of the involved system is only the *orbital* one $\vec{L}$ that has always univalence equal to + 1.

One could, at this point, think to adapt the proof of the theorem V.1 observing that given $j_{(1),1}, \cdots, j_{(1),Z}, j_{(2),1}, \cdots, j_{(2),Z} \in \frac{\mathbb{N}}{2}$ and such that:

$$\overleftarrow{|j_{(1)1}, m_{(1)1}, \cdots, j_{(1)Z}, m_{(1)Z}, j_{(2)1} := j_{(1)1}, m_{(2)1}, \cdots, j_{(2)Z} := j_{(1)Z}, m_{(2)Z} >} = - |j_{(1)1}, m_{(1)1}, \cdots, j_{(1)Z}, m_{(1)Z}, j_{(2)1} := j_{(1)1}, m_{(2)1}, \cdots, j_{(2)Z} := j_{(1)Z}, m_{(2)Z} > \quad (5.53)$$

the proposition IV.2 would imply that:

$$< j_{(1)1}, m_{(1)1}, \cdots, j_{(1)Z}, m_{(1)Z}, j_{(2)1} := j_{(1)1}, m_{(2)1}, \cdots, j_{(2)Z} := j_{(1)Z}, m_{(2)Z} \overleftarrow{|j_{(1)1}, m_{(1)1}, \cdots, j_{(1)Z}, m_{(1)Z}, j_{(2)1} := j_{(1)1}, m_{(2)1}, \cdots, j_{(2)Z} := j_{(1)Z}, m_{(2)Z} >} = 0 \quad (5.54)$$

that, combined with the equation 5.24 assuring that T leaves invariant any eigenspace of H, would allow to infer that each energy level is at least doubly degenerate.

Such an argument is not, anyway, correct since:

$$j_{(2)I} = j_{(1)I} \; \forall I \in \{1, \cdots, Z\} \; \Rightarrow [(-1)^{2 \sum_{I=1}^{Z} (j_{(1)I} + j_{(2)I})} \; = \; (-1)^{4 \sum_{I=1}^{Z} j_{(1)I}} = 1 \;\; \forall j_{(1)I} \in \frac{N}{2}, \forall I \in \{1, \cdots, Z\}] \quad (5.55)$$

and hence:

$$\overleftarrow{|j_{(1)1}, m_{(1)1}, \cdots, j_{(1)Z}, m_{(1)Z}, j_{(2)1} := j_{(1)1}, m_{(2)1}, \cdots, j_{(2)Z} := j_{(1)Z}, m_{(2)Z} >} = + |j_{(1)1}, m_{(1)1}, \cdots, j_{(1)Z}, m_{(1)Z}, j_{(2)1} := j_{(1)1}, m_{(2)1}, \cdots, j_{(2)Z} := j_{(1)Z}, m_{(2)Z} > \quad (5.56)$$

Example V.2

Let us now consider a situation identical to the one discussed in the example V.1 but for the fact that the $Z \in \mathbb{N} : n > 2$ involved particles are, this time, fermions of spin 1/2.

Introduced the total spin:

$$\vec{S} \; := \; \sum_{I=1}^{Z} \vec{S}_I \quad (5.57)$$

let us remark, first of all, that obviously:

$$[H, \vec{S}_{(I)}] \ = \ [H, S_{(I)3}] \ = \ [H, \vec{S}] \ = \ [H, S_3] \ = \ 0 \ \ \forall I \in \{1, \cdots, Z\} \tag{5.58}$$

Clearly:

$$H|j_{(1)1}, m_{(1)1}, \cdots, j_{(1)Z}, m_{(1)Z}, j_{(2)1} := j_{(1)1}, m_{(2)1}, \cdots, j_{(2)Z} := j_{(1)Z}, m_{(2)Z}, m_{(s)1}, \cdots, m_{(s)Z} > =$$
$$E_{j_{(1),1}, \cdots, j_{(1),Z}}|j_{(1)1}, m_{(1)1}, \cdots, j_{(1)Z}, m_{(1)Z}, j_{(2)1} := j_{(1)1}, m_{(2)1}, \cdots, j_{(2)Z} := j_{(1)Z}, m_{(2)Z}, m_{(s)1}, \cdots, m_{(s)Z} > \tag{5.59}$$

where $E_{j_{(1),1}, \cdots, j_{(1),Z}}$ is given again by the equation 5.34 and by the equation 5.51.

The degeneration of the energy levels, anyway, is this time given by

$$degeneration(E_{j_{(1),1}, \cdots, j_{(1),Z}}) \ = \ 2Z + 2 \prod_{I=1}^{Z} (2 j_{(1),I} + 1) \tag{5.60}$$

owing to the 2 possible values that each $m_{(s)I}$ can take.

Let us now observe that:

$$\overleftarrow{\overleftarrow{|j_{(1)1}, m_{(1)1}, \cdots, j_{(1)Z}, m_{(1)Z}, j_{(2)1} := j_{(1)1}, m_{(2)1}, \cdots, j_{(2)Z} := j_{(1)Z}, m_{(2)Z}, m_{(s)1}, \cdots, m_{(s)Z} >}} \ =$$
$$(-1)^{4 \sum_{I=1}^{Z} j_{(1)I}} (-1)^{Z} |j_{(1)1}, m_{(1)1}, \cdots, j_{(1)Z}, m_{(1)Z}, j_{(2)1} := j_{(1)1}, m_{(2)1}, \cdots, j_{(2)Z} := j_{(1)Z}, m_{(2)Z}, m_{(s)1}, \cdots, m_{(s)Z} > \tag{5.61}$$

and hence:

$$\overleftarrow{\overleftarrow{|j_{(1)1}, m_{(1)1}, \cdots, j_{(1)Z}, m_{(1)Z}, j_{(2)1} := j_{(1)1}, m_{(2)1}, \cdots, j_{(2)Z} := j_{(1)Z}, m_{(2)Z}, m_{(s)1}, \cdots, m_{(s)Z} >}} \ =$$
$$\begin{cases} +|j_{(1)1}, m_{(1)1}, \cdots, j_{(1)Z}, m_{(1)Z}, j_{(2)1} := j_{(1)1}, m_{(2)1}, \cdots, j_{(2)Z} := j_{(1)Z}, m_{(2)Z}, m_{(s)1}, \cdots, m_{(s)Z} >, & \text{if } Z \in \mathbb{E}; \\ -|j_{(1)1}, m_{(1)1}, \cdots, j_{(1)Z}, m_{(1)Z}, j_{(2)1} := j_{(1)1}, m_{(2)1}, \cdots, j_{(2)Z} := j_{(1)Z}, m_{(2)Z}, m_{(s)1}, \cdots, m_{(s)Z} >, & \text{if } Z \in \mathbb{O}. \end{cases} \tag{5.62}$$

If the number Z of involved fermions is even no information about the degeneracy of the energy levels can, consequentially, be inferred by the *double-time-reversal* symmetry of the hamiltonian.

Let us suppose, contrary, that $Z \in \mathbb{O}$.

Then, posed:

$$|\psi > := \ |j_{(1)1}, m_{(1)1}, \cdots, j_{(1)Z}, m_{(1)Z}, j_{(2)1} := j_{(1)1}, m_{(2)1}, \cdots, j_{(2)Z} := j_{(1)Z}, m_{(2)Z}, m_{(s)1}, \cdots, m_{(s)Z} > \tag{5.63}$$

the proposition IV.2 implies that:

$$< \psi | \overleftarrow{\psi >} \ = \ 0 \tag{5.64}$$

that, combined with the equation 5.24 assuring that T leaves invariant any eigenspace of H, allows to infer that each energy level is at least doubly degenerate.

Let us remark, anyway that such an inference is not an application of the theorem V.1 since it hasn't been obtained taking into account the univalence of the angular momentum $\vec{L} + \vec{S}$.

APPENDIX A: SOME USEFUL ALGEBRAIC PROPERTIES OF THE SET OF THE HALF-INTEGER NUMBERS

Obviously $(\frac{\mathbb{N}}{2}, +)$ is a commutative group.
Introduced:

- the *set of the even numbers*:

$$\mathbb{E} \;:=\; \{2n, n \in \mathbb{N}\} \tag{A1}$$

- the *set of the odd numbers*:

$$\mathbb{O} \;:=\; \{2n+1, n \in \mathbb{N}\} \tag{A2}$$

- the *set of the half-odd numbers*:

$$\mathbb{H} \;:=\; \frac{\mathbb{O}}{2} \tag{A3}$$

one has clearly that:

Proposition A.1

$$\frac{\mathbb{N}}{2} \;=\; \mathbb{N} \cup \mathbb{H} \tag{A4}$$

$$\mathbb{N} \cap \mathbb{H} \;=\; \emptyset \tag{A5}$$

$$x + y \in \mathbb{E} \;\; \forall x, y \in \mathbb{E} \tag{A6}$$

$$x + y \in \mathbb{O} \;\; \forall x \in \mathbb{E}, \forall y \in \mathbb{O} \tag{A7}$$

$$x + y \in \mathbb{E} \;\; \forall x \in \mathbb{O}, \forall y \in \mathbb{O} \tag{A8}$$

$$x + y \in \mathbb{H} \;\; \forall x \in \mathbb{N}, \forall y \in \mathbb{H} \tag{A9}$$

$$x + y \in \mathbb{N} \;\; \forall x, y \in \mathbb{H} \tag{A10}$$

$$|x - y| \in \mathbb{N} \;\; \forall x, y \in \mathbb{H} \tag{A11}$$

$$|x - y| \in \mathbb{H} \;\; \forall x \in \mathbb{N}, \forall y \in \mathbb{H} \tag{A12}$$

APPENDIX B: SOME META-TEXTUAL CONVENTION

I would like, first of all, to clarify that my use, in this as in any of my previous papers, of the first person plural has not to be considered as an act of arrogance (id est as a sort of *pluralis maiestatis)* but as its opposite: as an act of modesty performed in order to include the reader in the dissertation.

Then I would like to remark that the dates reported in the bibliography are those of the edition of the books that I have used.

This might lead some reader to misleading conclusions concerning the historical development of ideas: for instance the 3^{th} section "Perpetual motion" of the 6^{th} chapter "Statistical Entropy" of Oliver Penrose's book [18] historically predates Charles Bennett's later exorcism of Maxwell's demon based on the observation that, assuming the Landauer's Principle stating that a recursive function can be computed without increasing the entropy of the Universe if and only if it is injective, the erasure of the demon's memory cannot be performed without increasing the entropy of the Universe and that it is exactly the contribution to the thermodynamical balance of the erasure of demon's memory that preserves from the violation of the Second Principle of Thermodynamics.

Last but not least I advise the reader that in this paper it has been used a Unit System in which $\hbar = 1$.

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